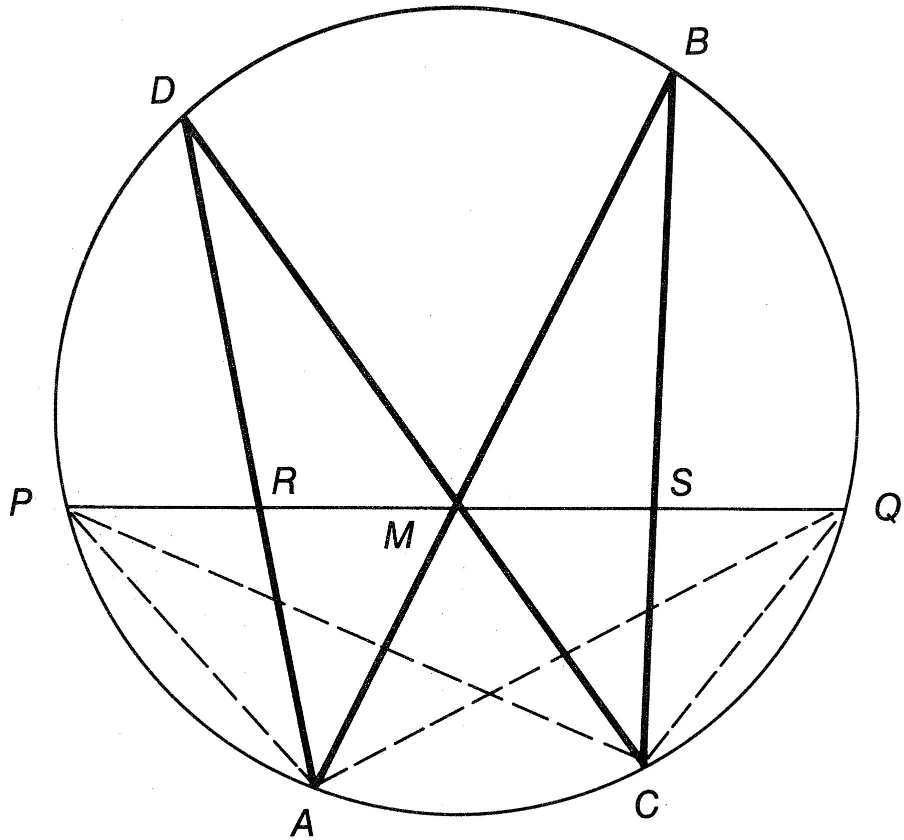
MATHEMATICS MAGAZINE



Vol. 60 No. 4
October 1987

BUTTERFLY PROBLEM • MATHEMATICS PLAY
GAMBLING SYSTEM • HARMONIC TRIANGLE

MATHEMATICS MAGAZINE

Volume 60, Number 4, October, 1987
(ISSN—0025-570X)

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EDITORIAL POLICY

The aim of *Mathematics Magazine* is to provide lively and appealing mathematical exposition. This is not a research journal and, in general, the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for an article for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Articles on pedagogy alone, unaccompanied by interesting mathematics, are not suitable. Neither are articles consisting mainly of computer programs unless these are essential to the presentation of some good mathematics. Manuscripts on history are especially welcome, as are those showing relationships between various branches of mathematics and between mathematics and other disciplines.

The full statement of editorial policy appears in this *Magazine*, Vol. 54, pp. 44–45, and is available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

Send new manuscripts to: G. L. Alexanderson, Editor, Mathematics Magazine, Santa Clara University, Santa Clara, CA 95053. Manuscripts should be typewritten and double spaced and prepared in a style consistent with the format of *Mathematics Magazine*. Authors should submit the original and one copy and keep one copy. Illustrations should be carefully prepared on separate sheets in black ink, the original without lettering and two copies with lettering added.

AUTHORS

Leon Bankoff received his B.S. degree in 1928 from the College of the City of New York and, in 1932, his D.D.S. degree from the University of Southern California School of Dentistry. Although an inveterate polymath, mathematics has always loomed large in his varied interests, with number theory, inequalities and advanced Euclidean and analytic geometry absorbing most of his attention. As problem editor of the *Pi Mu Epsilon Journal* for 15 years, he became immersed in a broad spectrum of mathematical topics, thereby being encouraged to delve into unfamiliar areas simply to stay ahead of the class. He has been a frequent contributor of problem proposals, solutions, and articles in numerous journals at home and abroad.

His paper, "The Metamorphosis of the Butterfly Problem," which appears in the current issue of this journal, represents the culmination of long research on multiple solutions of classic mathematical problems. As a dentist of long standing, now on the verge of retirement, he hopes finally to find the leisure to sit down and complete a long dormant manuscript begun thirty years ago in collaboration with Victor Thébault, embracing original modern extensions of the Arbelos, otherwise known as the Shoemaker's Knife of Archimedes, and introducing the application of inversion in the search for new properties.

Credit. Dave Logothetti provided the excellent artwork for the Bankoff article on the Butterfly problem.

Acknowledgment. The filler on p. 242 was kindly provided by Joel Zeitlin of California State University, Northridge.

The *Mathematics Magazine* (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Montpelier, VT, bimonthly except July/August.

The annual subscription price for the MATHEMATICS MAGAZINE to an individual member of the Association is $11 included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $22. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The non-member/library subscription price is $28 per year. Bulk subscriptions (5 or more copies) are available to colleges and universities for classroom distribution to undergraduate students at a 41% discount ($6.50 per copy–minimum order $32.50).

Subscription correspondence and notice of change of address should be sent to the Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20077-9564. Back issues may be purchased, when in print, from P. and H. Bliss Company, Middletown, CT 06457. Microfilmed issues may be obtained from University Microfilms International, Series Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

Advertising correspondence should be addressed to Ms. Elaine Pedreira, Advertising Manager, The Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036.

Second class postage paid at Washington, D.C. and additional mailing offices.

Postmaster: Send address changes to Mathematics Magazine Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

ARTICLES

The Metamorphosis of the Butterfly Problem

LEON BANKOFF
Los Angeles, CA 90048

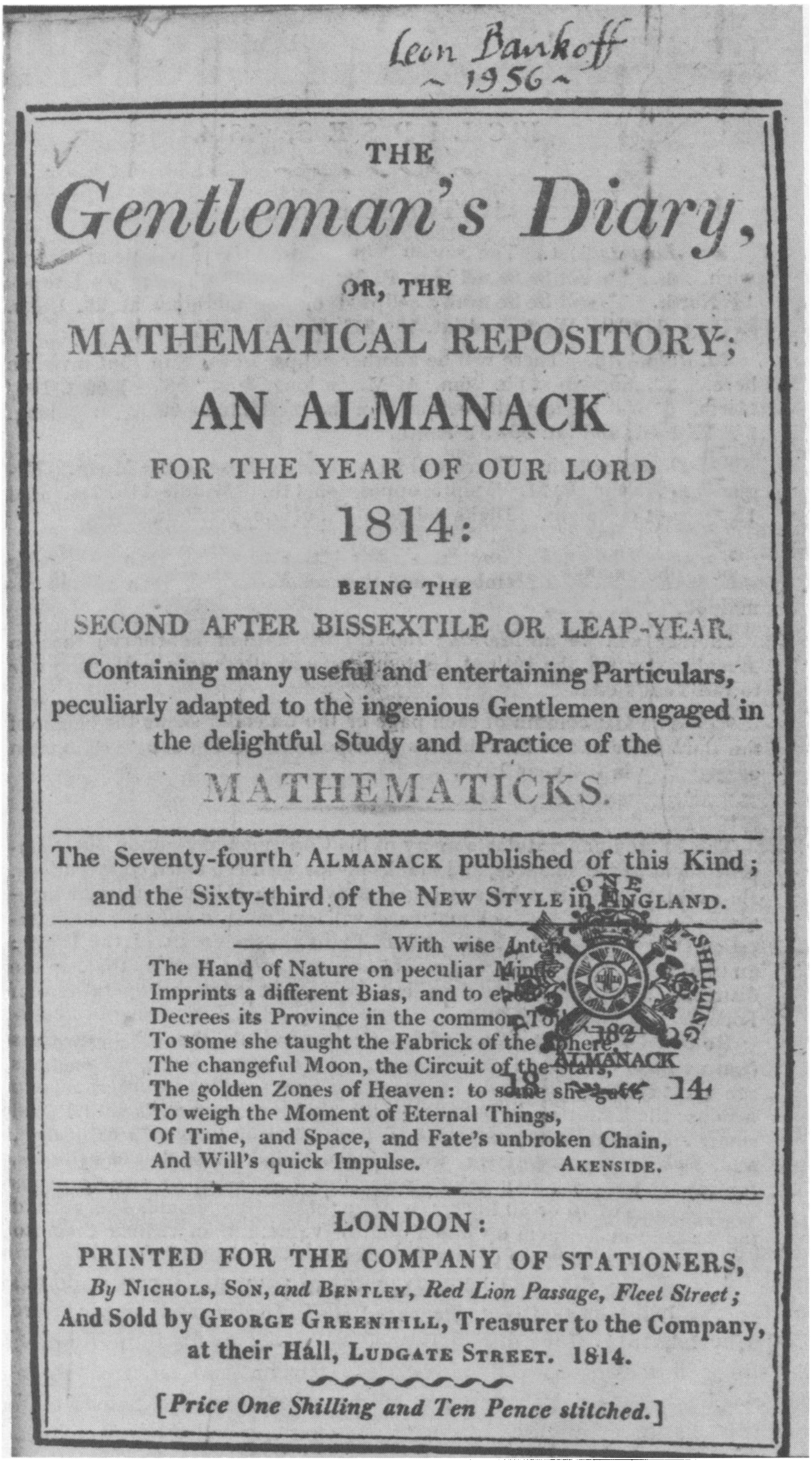
Editor's note. This article illustrates the diversity of geometric techniques that can be brought to bear on a single problem. The author was prompted to examine his ample collection of historical material when a compilation of varied proofs of the Butterfly problem was offered by Kaidy Tan of Fukien Teacher's University, Foochow, Fukien, China. All of these proofs had appeared in print, and this article outlines many of them, providing their historical roots.

One of the hardiest of the hardy perennials in the realm of Euclidean geometry is the problem that was dubbed The Butterfly by some as yet unidentified poetic mathematician who fancied the image of a lepidopterous creature in the configuration of the problem. This appellation made its first appearance as the title of solutions published in the *American Mathematical Monthly* in February 1944 [1]. The name took hold and has probably contributed to some extent to the recent popularity of the problem. My love affair with The Butterfly began thirty years ago with the publication of the following proposal in *School Science and Mathematics*:

> *In a circle* (O), P *is the midpoint of chord* AB. *Chords* RS *and* TV *pass through the point* P. RV *cuts* AP *at a point* M, *and* ST *cuts* PB *at point* N. *Prove by high school geometry that* MP *equals* PN.

I was fascinated by the problem, intrigued by the unexpected symmetry arising from an obviously random construction. Searching through my well-stocked library, I came across two solutions to the problem in the *Gentleman's Diary* of 1815 [2], a British publication that was instrumental in popularizing mathematics during the eighteenth and nineteenth centuries. I was pleasantly surprised to find that one of the solutions was by W.G. Horner (of Horner's method fame) who thought enough of the problem to submit a solution. I condensed his result and sent it along with mine for publication in SSM [3]. In a discussion of the problem with my good friend, guide, and mentor, Charles W. Trigg, I was fortunate enough to obtain several additional SSM references, plus the AMM reference mentioned above, as well as one in Johnson's *Modern Geometry* [4]. I was amazed to find a fantastic variety of solutions in these sources—some by Desargues' theorem on involution, some by the use of cross ratios, others stemming from Menelaus, analytic geometry, trigonometry, advanced Euclidean geometry, and various other modalities, all to be treated later on.

I wish to offer here samples of the different modes of attack on the Butterfly problem and to retrace my adventure in tracking down its history, its extensions and variations to date. It is my hope to enlarge upon the extensive treatment of the subject by Léo Sauvé, editor of *Crux Mathematicorum*, in his survey published in what was formerly called *Eureka*, in 1976 [5]. My investigations, though sporadic, were made all the more difficult by the namelessness of the problem before 1944. Rather than attempt a more systematic compilation, I shall cover these topics in more or less chronological order of discovery. Such loose, anecdotal treatment will enable the reader to join me along the path from idle curiosity to gratified revelation.



Leon Bankoff
~ 1956 ~

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OR, THE

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BEING THE

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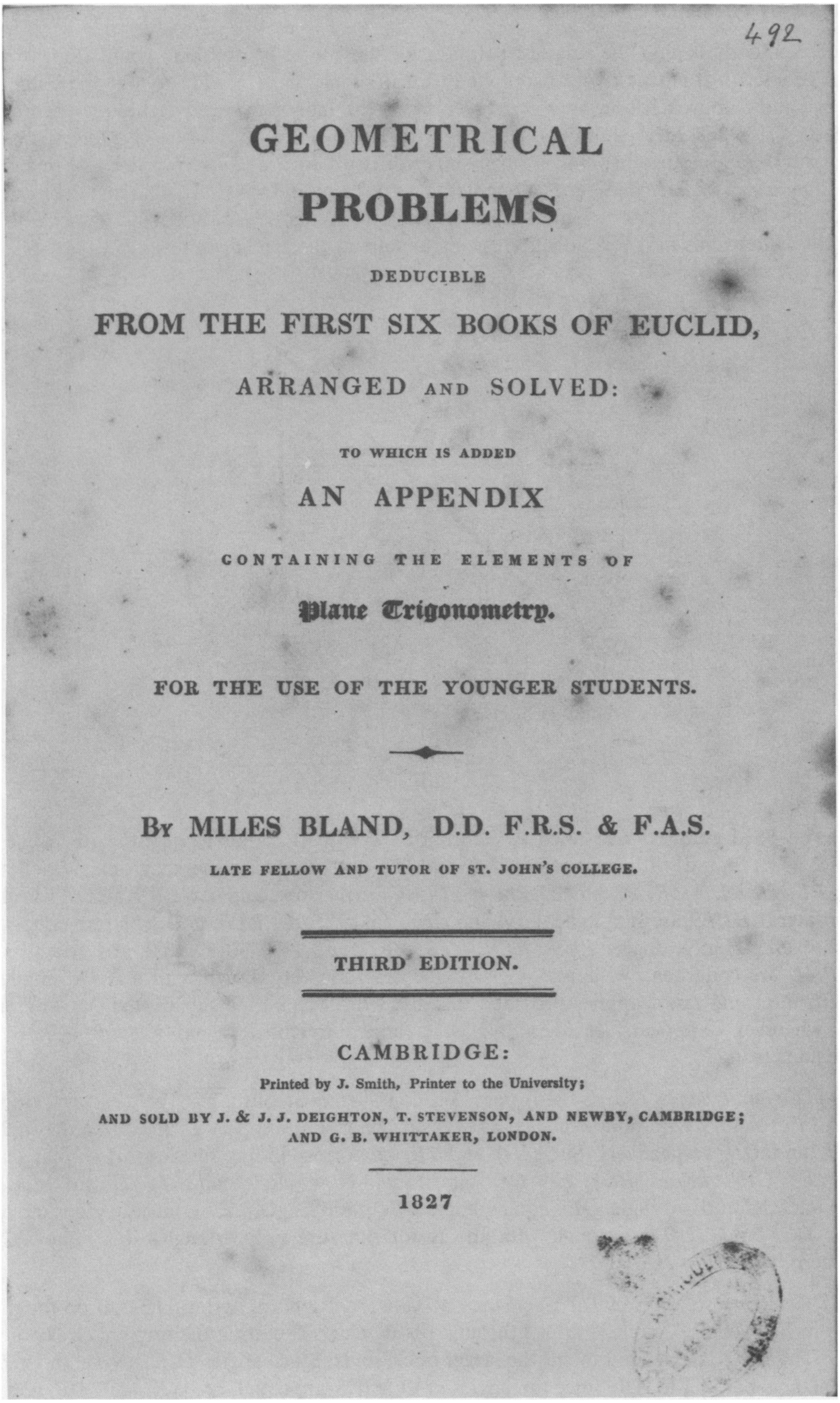

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GEOMETRICAL

PROBLEMS

DEDUCIBLE

FROM THE FIRST SIX BOOKS OF EUCLID,

ARRANGED AND SOLVED:

TO WHICH IS ADDED

AN APPENDIX

CONTAINING THE ELEMENTS OF

Plane Trigonometry.

FOR THE USE OF THE YOUNGER STUDENTS.

BY MILES BLAND, D.D. F.R.S. & F.A.S.

LATE FELLOW AND TUTOR OF ST. JOHN'S COLLEGE.

THIRD EDITION.

CAMBRIDGE:

Printed by J. Smith, Printer to the University;

AND SOLD BY J. & J. J. DEIGHTON, T. STEVENSON, AND NEWBY, CAMBRIDGE;
AND G. B. WHITTAKER, LONDON.

1827

Solutions by elementary geometry

One would think that the simplest solution for the Butterfly problem would stem from high school geometry. It is true that a neophyte in Euclid could more readily follow and comprehend an elementary approach but, as will be seen later, the more advanced techniques of projective geometry offer delightful shortcuts. According to Roger Johnson, author of *Modern Geometry*, "This simple appearing theorem is surprisingly difficult to prove." Howard Eves, in his book, *Fundamentals of Geometry* [6], goes along with this opinion and says, "It is a real stickler if one is limited to the use of only high school geometry." Whether or not you agree with these evaluations, it seems best for didactic reasons to start at the Euclidean level.

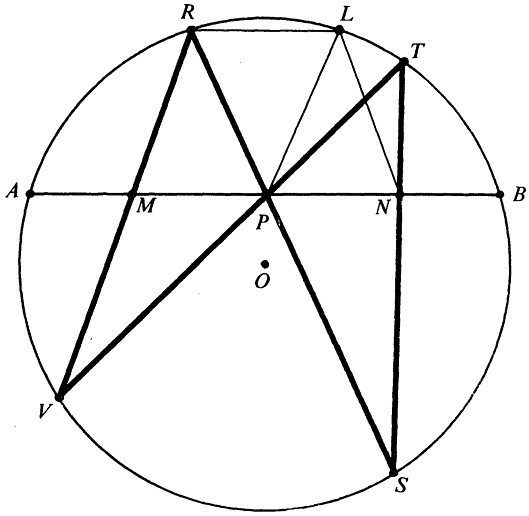


FIGURE 1

1) The SSM solution published in February 1955 [3] (FIGURE 1) begins with a chord RL parallel to AB and with connectors LP and LN as shown in the diagram. We then have $RP = PL$, $\angle PRL = \angle RLP$, and $\angle RPM = \angle LPN$. Furthermore, angles LRS and LTS in cyclic quadrilateral $RLTS$ are supplementary, and since angles LRS, RLP, and LPN are equal, $PNTL$ is also cyclic. Hence, angles PLN, PTN, VTS, VRS, and MRP are equal, and triangles MPR and PNL are congruent, with $MP = PN$. If the diagram were drawn with arc AR smaller than arc TB, the same proof applies provided that the word "equal" is substituted for "supplementary" wherever it appears. Incidentally, as an added touch, RT and VS cut AB at points equidistant from P.

2) *The Gentleman's Diary*, 1815, page 39, contains two solutions. The first, by Horner, is shown in FIGURE 2 with points labeled as in his original solution. Let D and E be the midpoints of ML and ON, respectively. Since ID and IE are corresponding medians of similar triangles MLI and CIN, angles PDI and QEI are equal. Now $DCIP$ and $ICEQ$ are each cyclic quadrilaterals, and so angle QEI equals angle ICQ, and angle PDI equals angle PCI. Hence, angles PCI and ICQ are equal, with the result that the right triangles ICP and ICQ are congruent, with $IP = IQ$.

3) The second solution in *The Gentleman's Diary*, by Richard Taylor, is based on the diagram shown in FIGURE 3. A circle is drawn through points Q, O, I, cutting the original circle at both O and S. Then SQ is extended to cut the larger circle in D. Since angles OIQ, OSD, and OLD are equal, DL is parallel to AB. Since the angles at O and S are equal, arc NL = arc DF, and NF is parallel to both DL and AB. Inasmuch as I lies on the perpendicular bisector of NF, it follows

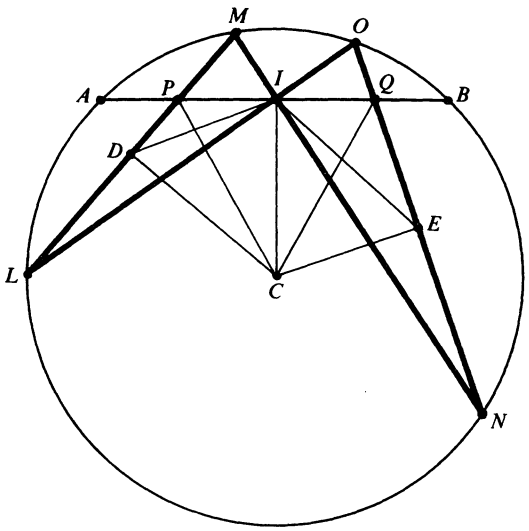


FIGURE 2

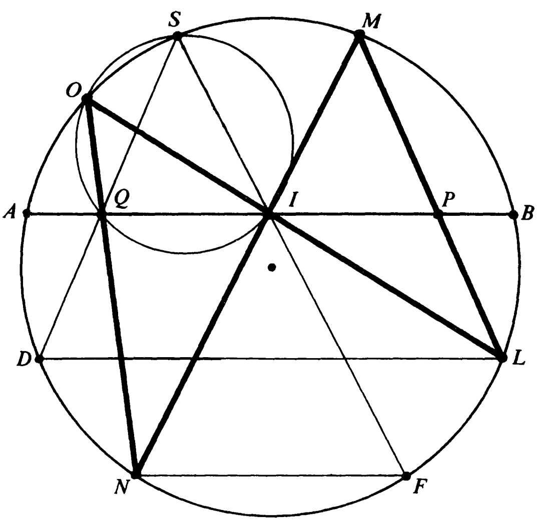


FIGURE 3

that $IN = IF$. Then with $\angle SNM = \angle SFM$ and with $\angle SIN = \angle MIF$, triangles SNI and MFI are congruent and $IS = IM$. Lastly, in congruent triangles ISQ and IPM, $QI = IP$; hence, $QA = PB$, and $AP = BQ$. The solver remarks that his demonstration applies where P and Q fall outside the circle. The latter version is proved in the *Educational Times*, problem 1549 [7] and in [**8**].

4) One of the early solutions of The Butterfly is found in another rare book, *Geometrical Problems*, by Miles Bland [**9**]. In FIGURE 4, C is the midpoint of AB and I, H are the respective

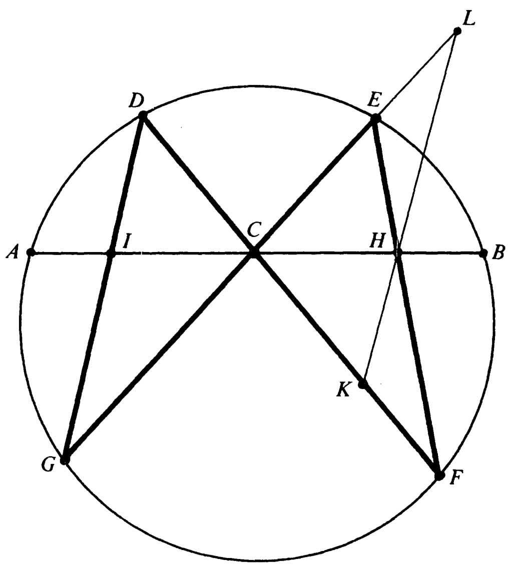


FIGURE 4

intersections of GD and FE with AB. KHL is drawn parallel to DG, meeting DF in K and GE (produced) in L. Since angles HLE, CGI, and HFK are equal, triangles LEH and HKF are equiangular and $LH/HE = HF/HK$. So $HE \cdot HF = LH \cdot HK = AH \cdot HB = (AC + CH)(AC - CH)$. Similarly, the triangles CID and CHK are equiangular, as are the triangles CHL and CIG.

Hence, $KH/HC = DI/IC$ and $LH/HC = GI/IC$. Then $KH \cdot LH/HC^2 = DI \cdot IC/IC^2$. But $KH \cdot LH = AC^2 - HC^2$ and $DI \cdot IC = AC^2 - IC^2$. Therefore $AC^2/HC^2 = AC^2/IC^2$, and $HC = IC$.

Except for slight changes in notation, the same solution was offered by W. E. Buker as Solution II of problem E 571 in the *American Mathematical Monthly*, February 1944 [**1**] and also by Coxeter and Greitzer, in *Geometry Revisited* [**10**].

5) The solution by Joseph Rosenbaum (see FIGURE 5), in the same section of AMM as Buker's solution in Example 4 above, depends on the reflection $D'F'$ of DF in the diameter through O

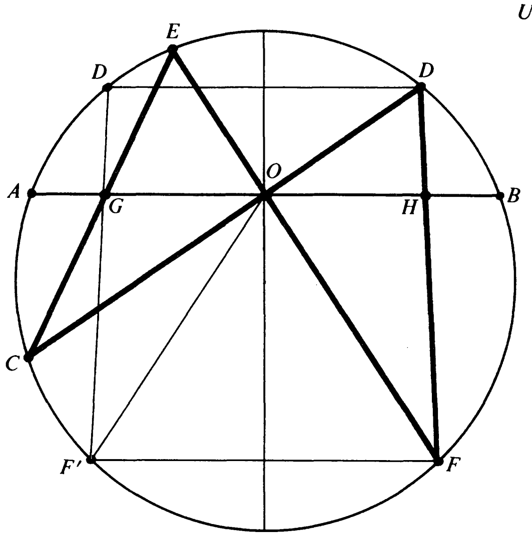


FIGURE 5

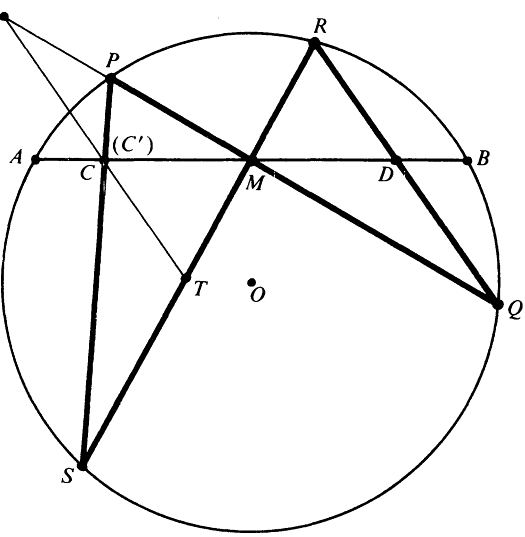


FIGURE 6

and the establishment of $OGCF'$ as concyclic, where G, H are the points of intersection of CE and DF with AB. Rosenbaum accomplishes this by the use of "directed angles" as set forth by Johnson in pages 11–15 of [**4**]. This is a useful, sophisticated concept but perhaps hard to grasp. For the sake of simplicity, I offer a paraphrased version, easily comprehended by anyone familiar with the first three books of Euclid's *Elements*.

By opposite vertical angles and by symmetry, the angles EOB, FOA, and BOF' are equal. Now, it is easily seen that angles ECF', EFF', and EOB are equal; hence $\angle ECF' = \angle BOF'$ ($= \angle GOF'$). It follows that O, G, C, F' are concyclic and that the angles $OF'G$, OCG, DCE, DFO, and $OF'D'$ are equal. Therefore, G lies on $D'F'$, and is the reflected image of H.

6) Mannis Charosh [**11**] offers an interesting approach based on a somewhat more advanced idea—one involving radical axes. Referring to the self-explanatory FIGURE 6, we first determine C' so that $MC' = MD$ and then prove that PS passes through C' thus making C and C' coincide. Let $MT = MR$ and $MU = MQ$. Then in the congruent triangles TUM and MQR, the similar triangles CUM and DQM are also congruent, whereupon $MC = MD$. Since angles S, Q, and U are equal, $PUST$ is a cyclic quadrilateral, whose circumscribing circle we call O'. In the given circle (O), points $ARBQ$ are concyclic. Hence points $BTAU$, the symmetric of $ARBQ$ with respect to M, are also concyclic. Call this circle O''. Now the radical axes AB, UT, PS of the three circles, (O), (O'), and (O''), are known to be concurrent. Finally, since AB and UT pass through C, PS must also pass through C. Hence $CM = MD$.

A similar solution by Charosh, also involving the intersection of three radical axes, was published in [**12**].

It might be tedious and unrewarding to wend one's way through all the many variations of these Euclidean proofs. One more example will suffice to illustrate the basic principle of solution, found repeatedly in the included list of related references.

7) The solution by Eilberg in SSM, January 1955 [**13**] starts with drawing the chord TY parallel to AB (FIGURE 7). Angle PYR (which is equal to the sum of angles PYT and TYR) is measured by half the arcs $AV + BT + TR$. Angle PMR is also measured by the same combination of arcs. Therefore, angles PMR and PYR are equal and $MPRY$ can be inscribed in a circle. So $\angle NTP = \angle MRP = \angle MYP$. Then triangles PYM and PTN are congruent, by ASA, since $PY = PT$ and $\angle APY = \angle NPT$, thus establishing the equality of PM and PN.

Anyone seeking to construct a new solution to The Butterfly could very well follow this basic sequence: reflect some element of the original figure, establishing a useful cyclic quadrilateral, and finally establish two congruent triangles containing the sides required to be proved equal. References containing similar routines, all derived from the basic pattern, are [**14**], [**15**], [**16**], [**17**], [**18**], [**19**], and [**20**].

Interesting variations of the Butterfly problem can be found in the following readily accessible references: [**5**], [**21**], [**22**], [**23**], [**24**], [**25**], and [**26**].

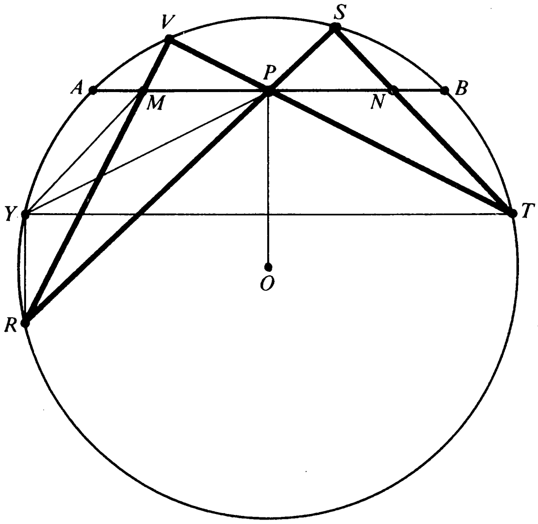


FIGURE 7

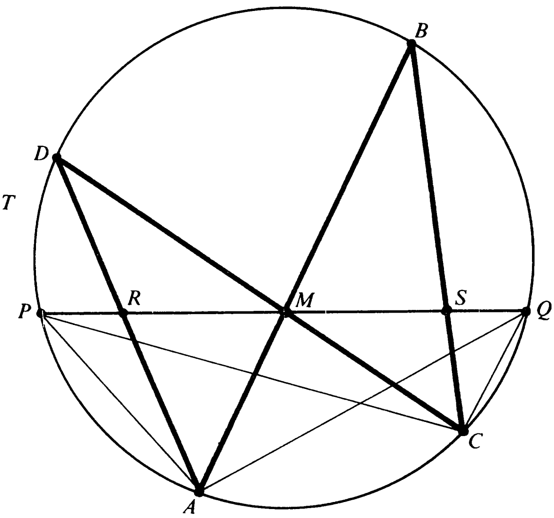


FIGURE 8

A solution by trigonometry

The treatment given in Miller's *College Geometry* [**27**] combines trigonometry with the use of cross ratios, also known as anharmonic ratios or double ratios. In FIGURE 8 the cross ratio $(PRMQ)$ is defined to be $\frac{PM}{MR}\Big/\frac{PQ}{QR}$. But

$$\frac{PM}{MR} = \frac{PM}{AM}\Big/\frac{MR}{AM} = \frac{\sin\angle PAM}{\sin\angle MPA}\Big/\frac{\sin\angle RAM}{\sin\angle MRA}.$$

Finding a similar quotient for $\frac{PQ}{QR}$ leads to

$$(PRMQ) = \frac{\sin\angle PAB}{\sin\angle BAD}\Big/\frac{\sin\angle PAQ}{\sin\angle QAD}.$$

In the same manner,

$$(PMSQ) = \frac{\sin\angle PCB}{\sin\angle BCD}\Big/\frac{\sin\angle PCQ}{\sin\angle QCD}.$$

Since angles in the same arc are equal, $(PRMQ) = (PMSQ)$ and

$$\frac{PM}{MR} \cdot \frac{QR}{PQ} = \frac{PS}{SM} \cdot \frac{QM}{PQ}.$$

Since $PM = MQ$, it follows that $RQ \cdot MS = PS \cdot RM$, which reduces to $RM \cdot MS + MQ \cdot MS = PM \cdot RM + MS \cdot RM$, or $MQ \cdot MS = PM \cdot RM$. Since $MQ = PN$, $MS = RM$.

This solution also appears in the article by Dixon Jones entitled "A Double Butterfly Theorem" [**25**].

Solution by analytic geometry

One way of misusing Descartes' powerful analytic geometry is to set up a rectangular coordinate system with the first chord of the circle falling along the x-axis and the diameter through the midpoint of that chord falling along the y-axis. One can write the equation of the circle and then find where the chords through the origin intersect the circle. The equations of the lines joining the intercepts lying on the circle are used to find the intercepts on the x-axis. After a bit of difficult manipulation, the sought-for intercepts on the x-axis are found to be equidistant from the origin. (This is a nasty method, recommended only for mathematicians with masochistic tendencies.)

A far more reasonable application of analytic geometry can be found in "A Simple Proof of the Butterfly Problem," by K. Satyanarayana [**28**]. The author sets up his initial configuration as in the previous paragraph but then introduces the notion of degenerate conics to prove not only the classic Butterfly Theorem for the circle, but also the generalized version for proper conics. His treatment as published in *Crux Mathematicorum* is so concise and precise that it needs no further abridgement and is offered here practically verbatim (with the kind permission of the editor Léo Sauvé). (FIGURES 9 and 9a.)

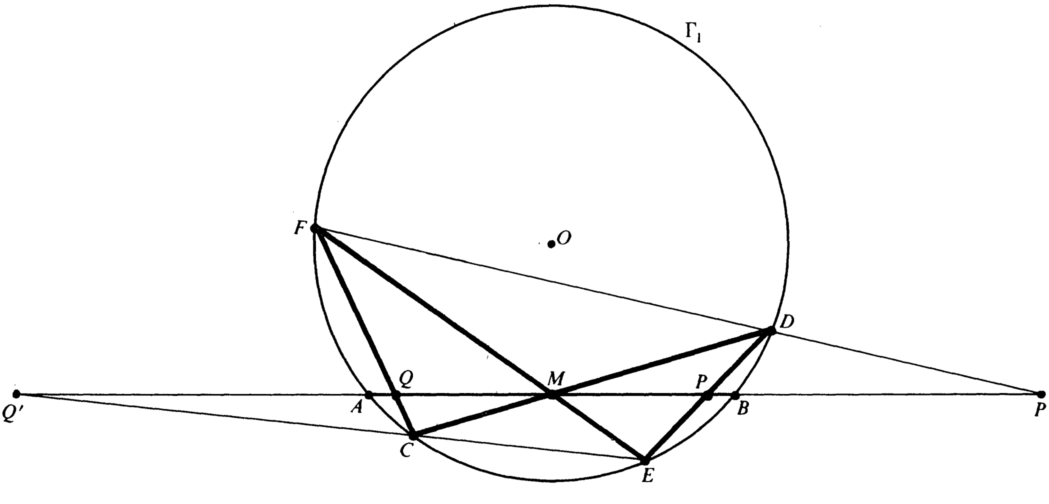


FIGURE 9

"THE BUTTERFLY PROBLEM. Through the midpoint M of a chord AB of a circle, two other chords, CD and EF, are drawn. ED and CF intersect AB in P and Q, respectively. Prove that $PM = MQ$.

Proof. Let Γ_1 be the given circle. We introduce a rectangular coordinate system with origin M, x-axis AB, and y-axis MO, where $O(0, d)$ is the centre of the circle. If the circle has radius r, its equation is

$$\Sigma_1 \equiv x^2 + (y-d)^2 - r^2 = 0.$$

As the lines CD and EF pass through the origin, they form a degenerate conic Γ_2 whose equation is of the form

$$\Sigma_2 \equiv ax^2 + 2hxy + by^2 = 0.$$

Now, for any k, l,

$$\Sigma \equiv k\Sigma_1 + l\Sigma_2 = 0$$

represents a conic Γ through the points common to Γ_1 and Γ_2, that is, through C, D, E, F; and every conic through C, D, E, F is representable in this form.

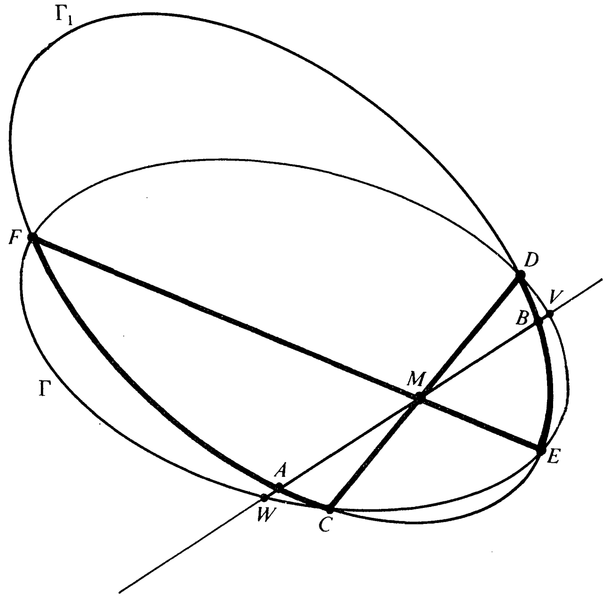


FIGURE 9a

Suppose the conic $\Sigma = 0$ intersects AB in V and W. The equation of AB is $y = 0$, and

$$\Sigma_1(x,0) = x^2 + d^2 - r^2, \qquad \Sigma_2(x,0) = ax^2;$$

hence the abscissas of V and W are the roots of $\Sigma(x,0) = 0$, that is, of

$$k(x^2 + d^2 - r^2) + lax^2 = 0.$$

Since this equation has no first-degree term, the sum of its roots is zero, so $\overline{MV} + \overline{MW} = 0$ and

$$VM = MW. \tag{1}$$

Now (1) holds for all conics through C, D, E, F, and the pair of lines ED, CF is such a conic, so $PM = MQ$ follows from (1). □

The pair of lines CE, DF is also a conic through C, D, E, F. If these lines intersect AB in P' and Q', as shown in FIGURE 1, then $P'M = MQ'$ also follows from (1).

Suppose that Γ_1 is, instead of a circle, an arbitrary *proper* conic of equation

$$\Sigma_1 \equiv Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0.$$

With the rest of the notation as before, we have

$$\Sigma(x,0) = k(Ax^2 + 2Gx + C) + lax^2.$$

If the coordinates of A and B are $(-\alpha, 0)$ and $(\alpha, 0)$, respectively, then $\Sigma_1(-\alpha,0) = \Sigma_1(\alpha,0) = 0$

implies that $G=0$; so the equation $\Sigma(x,0)=0$ has no first-degree term, and the rest of the proof is as before. We have thus proved

THE GENERALIZED BUTTERFLY PROBLEM. Through the midpoint M of a chord AB of a proper conic Γ_1, two other chords, CD and EF, are drawn. A conic Γ through C, D, E, F intersects AB in V and W. Prove that $VM=MW$."

This problem is illustrated in FIGURE 9a. A proof by projective geometry can be found in chapter 6 of Eves [**38**].

A solution by the theory of transversals

In 1919, *School Science and Mathematics* published a solution based on an application of Menelaus' Theorem [**29**]. Referring to FIGURE 10, consider the triangle HMK, cut by FG and DE. From the relations

$$\frac{CH}{CK}\cdot\frac{GK}{GM}\cdot\frac{FM}{FH}=1,\qquad \frac{CH}{CK}\cdot\frac{DK}{DM}\cdot\frac{EM}{EH}=1,$$

we obtain

$$\frac{CH^2\cdot GK\cdot DK\cdot FM\cdot EM}{CK^2\cdot FH\cdot EH\cdot GM\cdot DM}=1.$$

But $FM\cdot EM=GM\cdot DM$. Hence

$$\frac{CH^2}{CK^2}=\frac{FH\cdot EH}{GK\cdot DK}=\frac{AH\cdot BH}{AK\cdot BK}=\frac{(AC+CH)(AC-CH)}{(AC+CK)(AC-CK)}=\frac{AC^2-CH^2}{AC^2-CK^2}=1,$$

from which it is easily seen that $CH=CK$.

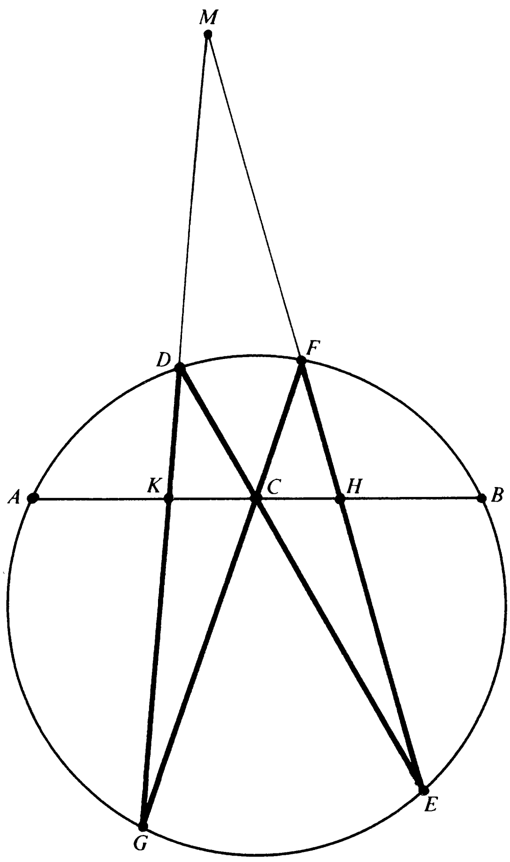


FIGURE 10

This solution is identical with the one published in 1907, in the French language text, *Exercices de Géométrie* [**30**].

Solutions by Desargues' theorem of involution

The solution of the Butterfly problem by involution is a striking example of the power and simplicity of projective geometry. Recall that an involution is a projectivity of period 2, that is, it exchanges pairs of points. A powerful theorem on involution, credited to Desargues, states that *any straight line* (*not a tangent*) *meets a conic and the opposite sides of an inscribed quadrangle in pairs of points in an involution*. In particular, for a cyclic quadrilateral, the points of intersection of a straight line with the three pairs of opposite sides of the complete quadrangle and with the circumscribing circle form four pairs of an involution. (For discussion and proofs of Desargues' theorems on involution for conics, consult any of the references numbered [**38**] to [**47**].)

School Science and Mathematics, March, 1919 [**29**], offers the following solution of the Butterfly problem by Philomathe, of Montreal, Canada:

AB is a chord in a given circle, bisected in C. DE and FG are any two chords intersecting each other in C; FE intersects AB in H, and DG intersects AB in K. Prove that $CH = CK$. (FIGURE 10).

Consider the cyclic quadrilateral $FDGE$ and its diagonals FG, DE. According to Desargues' theorem on involution, we have

$$\frac{AH \cdot AK}{AC \cdot AC} = \frac{BH \cdot BK}{BC \cdot BC}.$$

Therefore $AH \cdot AK = BH \cdot BK$ or $AH/BK = BH/AK = (AH + BH)/(BK + AK) = 1$. Hence, $AH = BK$ and $CH = CK$.

A practically identical solution is given in [**30**]. All of our notions of involution, the invariance of cross ratios, poles and polars, reciprocation, perspectivity, and the countless corollaries arising therefrom can be traced back to Desargues. It is safe to say that our current versions of the Butterfly are extensions of Desargues' work on conics and not the other way around. According to Coolidge [**31**], it would be even more proper to recognize that many of Desargues' ideas should be credited to Pappus [**32**]. One of my illustrious colleagues remarks that the germs of the Butterfly Theorem began with Pappus and then crawled along at a caterpillar pace until the larvae were completely encased by Desargues in the 17th century. The pupae were enclosed in a silken threaded cocoon and did not reach adulthood as butterflies until released from quiescence two centuries later by Chasles and other projective geometers.

Another dramatic example of the efficacy of projective methods is given next.

A solution by the use of cross ratios

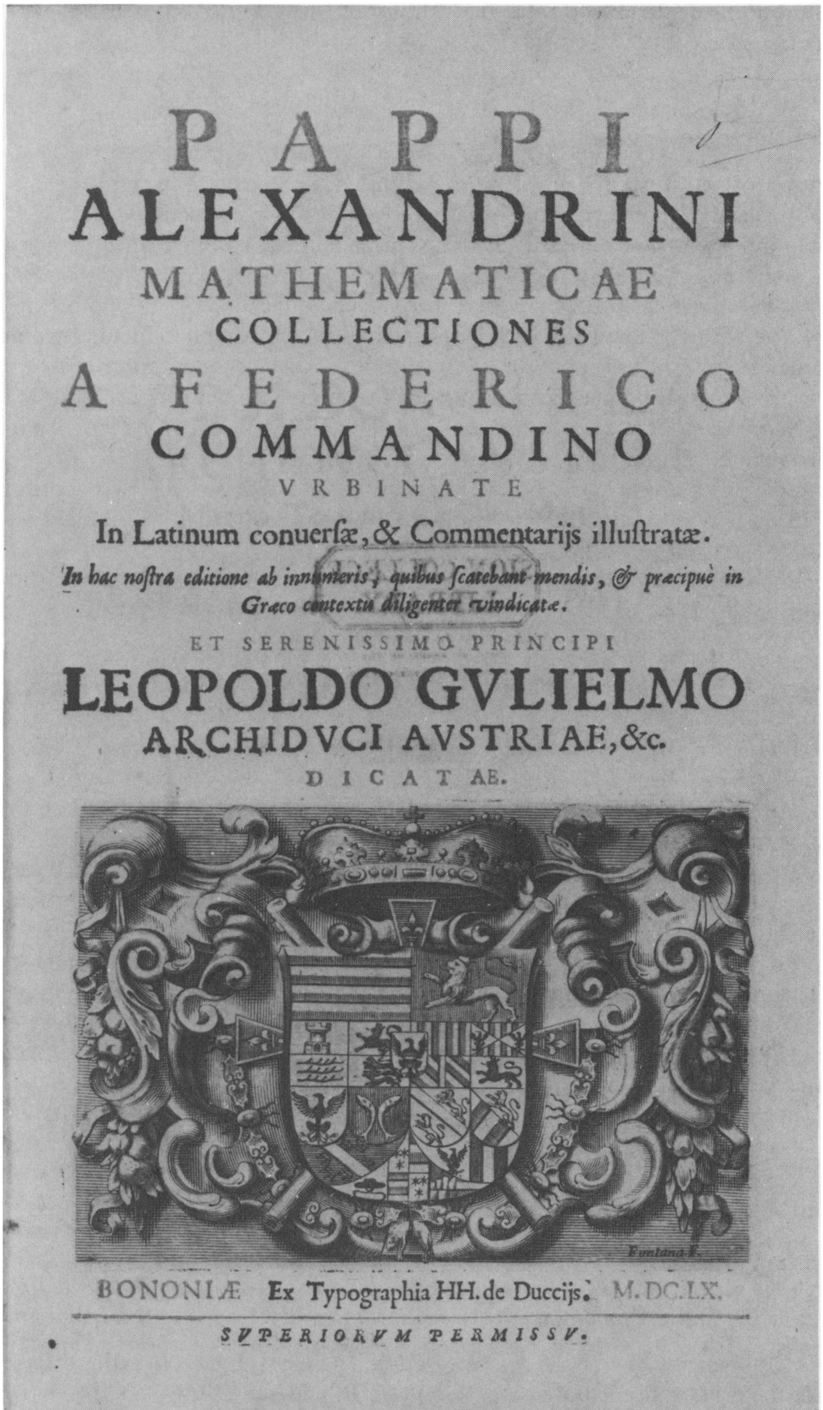
John Casey's *Sequel to Euclid* [**33**] contains an elegant disposition of the Butterfly problem as follows:

If through the middle point O of any chord AB of a circle two other chords CE and DF are drawn, and if the lines ED and CF joining their extremities intersect AB in G and H, then $OG = OH$.

Solution. The pencil $(E \cdot ADCB) = (F \cdot ADCH)$. Therefore, the cross ratio of the points A, G, O, B equals the cross ratio of the points A, O, H, B, and since $AO = OB, OG = OH$.

The foregoing solution is Casey's verbatim. In more familiar notation, the vertex of the pencil would lie outside the bracketed range.

Once the principles behind cross ratios are understood, it is not necessary to derive them each time they are used, any more than one would have to repeat the proof of the Pythagorean theorem every time it appears in a solution. Consequently we have in cross ratio a ready made shortcut for countless applications.



PAPPI
ALEXANDRINI
MATHEMATICAE
COLLECTIONES
A FEDERICO
COMMANDINO
VRBINATE

In Latinum conuersæ, & Commentarijs illustratæ.

In hac nostra editione ab innumeris, quibus scatebant mendis, & præcipuè in Græco contextu diligenter vindicatæ.

ET SERENISSIMO PRINCIPI
LEOPOLDO GVLIELMO
ARCHIDVCI AVSTRIAE, &c.
DICATAE.

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A remarkable theorem

A footnote in Roger Johnson's *Modern Geometry* [4] calls attention to a remarkable article by A. Candy published in *Annals of Mathematics* in 1896 [34]. Because of the relative inaccessibility of the publication, the material pertaining to our main theme of The Butterfly is given here, paraphrased for easier comprehension.

A point O is taken **anywhere** *on a chord AB of a circle and chords CD and EF are drawn through O. Then chords CF and ED cut AB in G and H, respectively* (FIGURE 11).

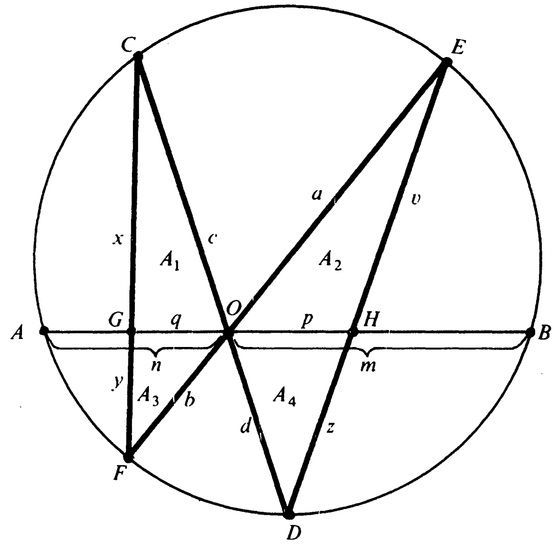

FIGURE 11

Let A_1, A_2, A_3, A_4 denote the areas of triangles CGO, EOH, GOF, and ODH, respectively.
Let $OB = m$, $OA = n$, $OH = p$, $OG = q$, $OE = a$, $OF = b$, $OC = c$, $OD = d$, $CG = x$, $GF = y$, $EH = v$, and $HD = z$.
Then, since $\angle C = \angle E, \angle F = \angle D$, etc.,

$$\frac{A_1}{A_2} = \frac{cx}{av}, \qquad \frac{A_3}{A_4} = \frac{by}{dz},$$

$$\frac{A_1}{A_4} = \frac{cq}{dp}, \qquad \frac{A_3}{A_2} = \frac{bq}{ap}.$$

It follows that

$$\frac{A_1A_3}{A_2A_4} = \frac{bcq^2}{adp^2} = \frac{bcxy}{advz},$$

whereupon

$$\frac{q^2}{p^2} = \frac{xy}{vz} = \frac{AG \cdot GB}{AH \cdot HB} = \frac{(n-q)(m+q)}{(n+p)(m-p)}$$

$$= \frac{mn - q(m-n) - q^2}{mn - p(m-n) - p^2}.$$

From this,

$$mn(p-q) = pq(m-n), \tag{1}$$

so

$$\frac{mn}{pq} = \frac{m-n}{p-q},$$

which can be written

$$1/q - 1/p = 1/n - 1/m. \tag{2}$$

Therefore, the products of the segments intercepted by the circle and the two chords, respectively, are proportional to their difference (or sums).

The Butterfly is an easy consequence of this beautiful general result since the point O in the statement of Candy's Theorem can be taken as the midpoint of AB. In this case, $m - n = 0$, and from (1), $mn(p-q) = 0$. Then since $mn \neq 0$, $p = q$.

Candy's article goes on to consider other cases that do not concern us here. The solution given here for the special classic Butterfly case surfaced again in [**35**], reprinted in [**5**].

The result given in (2) is particularly useful for solving "The Three-Winged Butterfly Problem," proposed by R. S. Luthar in the March 1984 issue of this MAGAZINE, with two solutions published in the March 1985 issue [**36**]. The solution by Jordi Dou follows the cross ratio method (described above) from Casey's *Sequel to Euclid*. The second solution, by Tiberio, makes use of Haruki's lemma, part of Honsberger's article [**14**]. Haruki's lemma is the subject of our next section. Luthar's proposal reads as follows:

Let chord AB of circle O be trisected at C and D. Let P be any point on the circle other than A and B. Extend the lines PD and PC to intersect the circle in E and F, respectively. Extend the lines EC and FD to intersect the circle in G and H, respectively. Let GF and HE intersect AB in L and M, respectively. Prove that $AL = BM$ (FIGURE 12).

Using the notation and the relations in the proof of Candy's Theorem, we let $AC = n = 1$, $CB = m = 2$, $CD = p = 1$, whereupon $LC = q = 2/3$. The same application in the crossed quadrilateral $PFHE$ yields $DM = 2/3$. Hence $AL = MB$ since $AC = DB$.

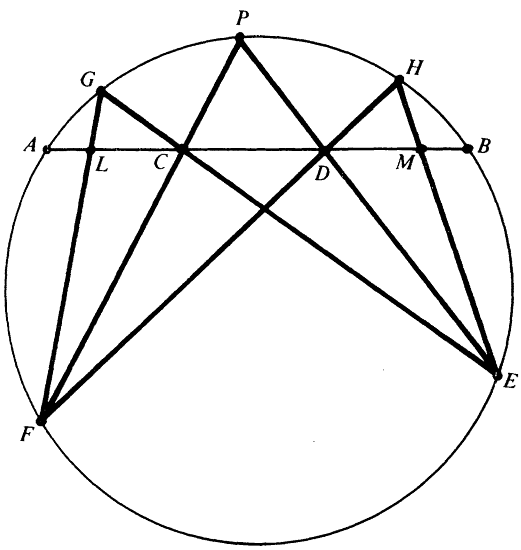

FIGURE 12

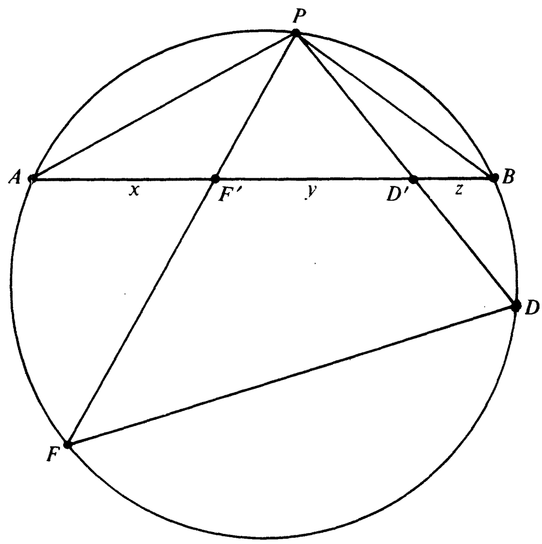

FIGURE 13

Haruki's Lemma

Haruki's Lemma, given in [**14**], states that *if AB and FD are nonintersecting chords in a circle and if P is a variable point on the arc AB remote from F and D, then for each position of P, the lines PF and PD cut AB into three segments of length x, y, z, where xz/y is constant* (FIGURE 13).

Haruki's proof is crisp and concise. An alternate proof by cross ratios is rather simple. Let $AB = m$ and draw PA and PB. Let AB cut PF in F' and PD in D'. Since the angles APF, FPD, DPB remain constant as P varies along arc AB (signified by the variable point P') the cross ratio of the pencil $P(AFDB)$ remains constant. Accordingly, the corresponding cross ratio $P(AF'D'B)$, or $P'(AF'D'B)$, also remains constant. We then have $F'A/F'D = BA/BD' = xz/ym$, or the more general $x'z'/y'm$.

Thus Haruki's Lemma is essentially the same as Chasles' Theorem: *If A, B, C, D are four fixed points of a conic and if P is a variable point of the conic, then the cross ratio $P(ABCD)$ is independent of the position of P on the conic.* (See **[37]**.)

The application of Chasles' Theorem, or Haruki's Lemma, to the Butterfly is self-evident. There is some particularly valuable editorial comment following the solution of "The Three-Winged Butterfly Problem" in **[36]**.

We owe our enjoyment of the Butterfly to the following celebrities, listed in retrograde order:

a) The nineteenth century luminaries who developed projective geometry: Chasles, Carnot, Poncelet, Gergonne, Von Staudt, Steiner, Monge and others.

b) Desargues.

c) Pappus.

d) Euclid, who, according to Chasles, was aware of the cross ratio property in his lost books on Porisms.

I am grateful to Dan Pedoe for the conversations we had regarding involution; to Léo Sauvé for supplying me with a copy of F.G.-M; to Murray Klamkin for catapulting me into the mainstream of mathematics; to Charles W. Trigg and to Solomon W. Golomb for our long friendship and ongoing inspiration; and to Archimedes, the most admirable streaker in history. I owe many thanks to editor-emeritus of *Mathematics Magazine*, Doris Schattschneider, for her patience and her valuable comments; also for her selection of a sharp-eyed referee who helped polish off some rough spots.

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It is (or used to be) well known that if you look in the index to Kerékjártó's *Topologie* for the name Bessel-Hagen, you find, on the indicated page, a collection of sketches of manifolds, one of which looks like a pumpkin-head with big ears. As an addition, I have been told that when the book was in proof Kerékjártó showed the picture to Courant (who was the editor) and said, "Doesn't this look like Bessel-Hagen?" Courant made some sort of noncommital reply that indicated "Yes, maybe it does." Then, when the book came out, Courant was in no position to object.

There is another joke in the book, less well known. If you look up "Kerékjártó," you find a single page reference; you look down the page without finding the name, and at the very bottom there is a footnote that says (freely translated) "This incorrect statement was made by me."

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Rank According to Perron: A New Insight

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If $A = (a_{ij})$, $a_{ij} > 0$, $i, j = 1, \ldots, n$, Perron [4] proved that A has a real positive eigenvalue $\lambda_{\max}$ (called the principal eigenvalue of A) that is unique, and $\lambda_{\max} > |\lambda_k|$ for the remaining eigenvalues of A. Furthermore, the principal eigenvector $w = (w_1, \ldots, w_n)$ that is a solution of $Aw = \lambda_{\max} w$ is unique to within a multiplicative constant and $w_i > 0$, $i = 1, \ldots, n$. We can make the solution w unique through normalization. We define the norm of the vector w as $\|w\| = we$ where $e = (1, 1, \ldots, 1)^T$, e^T is its transpose, and to normalize w is to divide it by its norm. We shall always think of w in normalized form.

Perron's result has found wide use in many areas, both theoretical and applied. Among these are applications to multicriterion decisions. In a recent article in this journal [1], Barbeau gave a lucid exposition and illustrated the use of the Analytic Hierarchy Process (AHP) to assist a college in interpreting grades and testing students for admission. The AHP, which I developed, uses Perron's principal eigenvector as an essential property for determining the preference ranking among a set of alternatives when the judgments are inconsistent. In the AHP the alternatives are compared in pairs with respect to a common attribute in a positive reciprocal matrix whose entries represent the numerical value of the relative preference for one (the row) over another (the column). The reader is referred to Barbeau's paper for a concise reference to the AHP. My purpose in this note is to show how important the principal eigenvector is in determining the rank of the alternatives through dominance walks.

Suppose we wish to rate five teachers A, B, C, D, and E according to their excellence in teaching. We enter our evaluation in the following matrix, whose principal eigenvector has been normalized:

	A	B	C	D	E	Eigenvector solution
A	1	1/6	1/2	1/9	5	.0893
B	6	1	2	1	5	.3287
C	2	1/2	1	1	5	.1983
D	9	1	1	1	5	.3413
E	1/5	1/5	1/5	1/5	1	.0424

This matrix gives the pairwise dominance of the alternative in the row over that in the column. For example, teacher B is rated to be 2 times better than teacher C, which is the entry in the $(2,3)$ position. The reciprocal value of $1/2$ is then entered in the transpose position which here is $(3,2)$. From this pairwise comparison matrix we wish to derive a scale of relative standing for the teachers. At first one may think that this is given by adding the components of each row and normalizing the result. This is only true if the matrix is consistent $(a_{ij}a_{jk} = a_{ik},\ i, j, k = 1, \ldots, n)$ for then the matrix has unit rank and any row is a multiple of a single row. The above matrix is inconsistent. For example, $a_{23} = 2 \neq a_{13}/a_{12} = (1/2)/(1/6) = 3$. Note that consistency implies the reciprocal relation but not conversely. Let us now examine the general case.

There is a natural way to derive the rank order of a set of alternatives from a pairwise comparison matrix A [**6**]. The rank order of each alternative is the relative proportion of its dominance over the other alternatives. This is obtained by adding the elements in each row in A and dividing by the total over all the rows. However, A only captures the dominance of one alternative over each other in one step. But an alternative can dominate a second by first dominating a third alternative, and then the third dominates the second. Thus, the first alternative dominates the second in two steps. It is known that the result for dominance in two steps is obtained by squaring the pairwise comparison matrix. Similarly, dominance can occur in three steps, four steps, and so on, the value of each obtained by raising the matrix to the corresponding power. The rank order of an alternative is the sum of the relative values for dominance in its row, in one step, two steps, and so on averaged over the number of steps. The question is whether this average tends to a meaningful limit.

We can think of the alternatives as the nodes of a directed graph. With every directed arc from node i to node j (which need not be distinct) is associated a nonnegative number a of the dominance matrix. In graph-theoretic terms this is the intensity of the arc. Define a k-walk to be a sequence of k arcs such that the terminating node of each arc except the last is the source node of the arc which succeeds it. The *intensity of a k-walk* is the product of the intensities of the arcs in the walk. With these ideas, we can interpret the matrix A: the (i, j) entry of A is the sum of the intensities of all k-walks from node i to node j [**2**, p. 203].

Definition. The dominance of an alternative along all walks of length $k \geqslant m$ is given by

$$\frac{1}{m}\sum_{k=1}^{m}\frac{A^k e}{e^T A^k e}.$$

Observe that the entries of $A^k e$ are the row sums of A^k and that $e^T A^k e$ is the sum of all the entries of A.

THEOREM. *The dominance of each alternative along all walks k, as $k \to \infty$, is given by the solution of the eigenvalue problem $Aw = \lambda_{\max} w$.*

Proof. Let

$$s_k = \frac{A^k e}{e^T A^k e},$$

and

$$t_m = \frac{1}{m}\sum_{k=1}^{m} s_k .$$

The convergence of the components of t_m to the same limits as the components of s_m is the standard Cesaro summability and is shown in Hardy [3]. Since

$$s_k = \frac{A^k e}{e^T A^k e} \to w \text{ as } m \to \infty$$

[**5**, p. 171] where w is the normalized principal right eigenvector of A, we have

$$t_m = \frac{1}{m}\sum_{k=1}^{m}\frac{A^k e}{e^T A^k e} \to w \text{ as } k \to \infty .$$

The essence of the principal eigenvector is to rank alternatives according to dominance in terms of walks. The well-known logarithmic least squares method (LLSM): find the vector $v = (v_1, \dots, v_n)$ which minimizes the expression

$$\sum_{i,j=1}^{n}\left(\log a_{ij} - \log\frac{v_i}{v_j}\right)^2,$$

sometimes proposed as an alternative method of solution, obtains results which coincide with the principal eigenvector for matrices of order two and three, but deviate from it for higher order and can lead to rank reversal. As an example of rank reversal, compare the eigenvector solution and the LLSM solution for the 5×5 matrix given earlier:

Teacher	Eigenvector solution	LLSM solution
A	.0893	.0819
B	.3287	.3433
C	.1983	.2089
D	.3413	.3214
E	.0424	.0418

In the eigenvector solution the teachers are ranked in descending order D, B, C, A, E; whereas the LLSM solution ranks them B, D, C, A, E.

The LLSM minimizes deviations over all the entries of the matrix. The principal eigenvector does not attempt to minimize anything, but maximizes information preserved from all known relations of dominance.

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Presenting a Mathematics Play

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Have you ever thought of presenting a mathematics play? Imre Lakatos' *Proofs and Refutations* can be easily understood by freshmen, yet contains interesting and even challenging material for all levels, including teachers and research mathematicians. The ideas he presents about how mathematics is formed and how it should be taught, his witty presentation, and his interest in the history of mathematics make it worth staging. The play is really a revised version of the first chapter of Lakatos' 1961 Cambridge Ph.D. thesis in the Philosophy of Mathematics, and it has been very carefully crafted. In it, he confronts the classical picture of mathematical development as a steady accumulation of established truths. He shows that mathematics grew instead through a richer, more dramatic process of successive improvements of creative hypotheses, by attempts to prove them, and by criticism of these attempts: by proofs and refutations.

Set in a classroom with one ultrapatient teacher and at least a dozen *very* bright students (who unwittingly draw their opinions from Euler, Legendre, Cauchy, Möbius, and others), the play is very lively with a rich cast of characters, from dogmatists to skeptics. The topic under discussion sometimes proposed as an alternative method of solution, obtains results which coincide with the principal eigenvector for matrices of order two and three, but deviate from it for higher order and can lead to rank reversal. As an example of rank reversal, compare the eigenvector solution and the LLSM solution for the 5×5 matrix given earlier:

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Have you ever thought of presenting a mathematics play? Imre Lakatos' *Proofs and Refutations* can be easily understood by freshmen, yet contains interesting and even challenging material for all levels, including teachers and research mathematicians. The ideas he presents about how mathematics is formed and how it should be taught, his witty presentation, and his interest in the history of mathematics make it worth staging. The play is really a revised version of the first chapter of Lakatos' 1961 Cambridge Ph.D. thesis in the Philosophy of Mathematics, and it has been very carefully crafted. In it, he confronts the classical picture of mathematical development as a steady accumulation of established truths. He shows that mathematics grew instead through a richer, more dramatic process of successive improvements of creative hypotheses, by attempts to prove them, and by criticism of these attempts: by proofs and refutations.

Set in a classroom with one ultrapatient teacher and at least a dozen *very* bright students (who unwittingly draw their opinions from Euler, Legendre, Cauchy, Möbius, and others), the play is very lively with a rich cast of characters, from dogmatists to skeptics. The topic under discussion is: "for which polyhedra does $V-E+F=2$?", where V, E, and F denote the numbers of vertices, edges, and faces. This innocuous question gives rise to a plethora of problems and subtopics, from the nature of a proof and mathematical truth to grounds for dismissal and ways of overcoming counterexamples.

It is possible to present the first forty pages (§1–§4) of the dialogue in an hour, with a minimum cast of eight by doubling up some roles. To avoid an actor ending up speaking to himself or even attacking his own theories, the following chart may help. It contains an X wherever such a conflict occurs in the first four sections and the approximate number of lines spoken by each character (Lakatos denotes students by Greek letters).

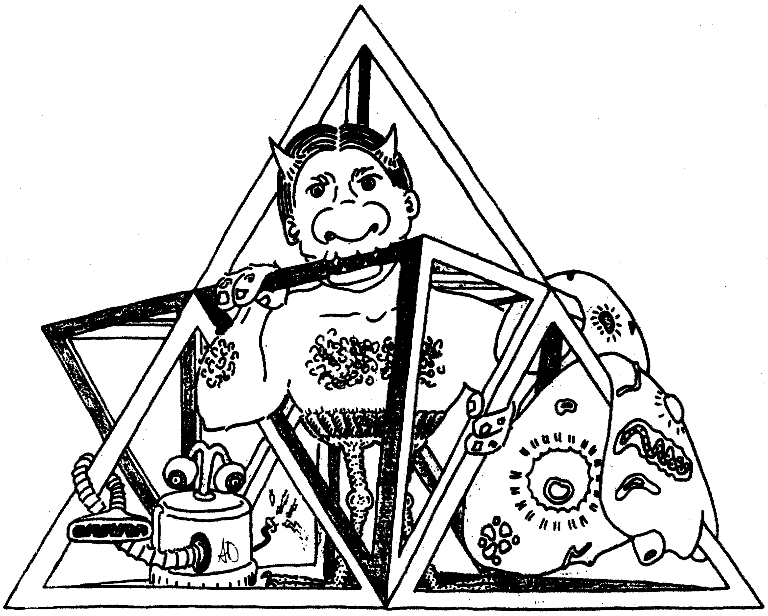
Lines	Character	ω	θ	ε	κ	η	ρ	σ	δ	γ	β	α	T
many	T												
142	α												X
129	β											X	X
111	γ										X	X	X
93	δ									X	X	X	X
30	σ								X	X	X	X	X
28	ρ							X	X	X	X	X	X
25	η						X	X	X		X	X	X
23	κ						X	X	X	X	X	X	X
5	ε				X			X	X	X	X	X	X
2	θ				X					X		X	X
6	ω												X
1	ι												X

A minimum cast would be Teacher, α, β, γ, $\delta=\theta=\iota$, $\sigma=\omega$, $\varepsilon=\rho$, and $\eta=\kappa$; of course, with more performers compound roles can be split.

Finding a suitable ending poses a slight problem, though. The last line of the excerpt §1–§4 has Gamma shouting that he has found a counterexample to the revised theorem (conjecture?). Leaving the audience on tenterhooks will ensure that they attend any subsequent excerpts, or at least will encourage them to read the play. Another ending could be written in, probably, or Gamma's last line could be deleted, but more reasonable would be to give a leisurely presentation ending after 50 minutes with the teacher's summary on pages 33–34.

The second author first participated in the play when David Fowler (University of Warwick, England) held a reading in his history of mathematics class in 1981. Twice a presentation has been organized, unrehearsed, with a minimum cast, in front of an audience in our own department. The performers and many members of the audience had dittoed copies of the script. The audience appreciated this because they could read the vital footnotes, which were not spoken. The cast sat at desks, set diagonally at the front of a regular classroom—their scripts blended with other papers on their desks. Although the pictures are few and easy to draw on a blackboard, we used an overhead projector for quick changes of predrawn diagrams during the class discussion. Students in the audience were particularly stimulated by the presentation: creative mathematics came alive and they better understood the link between their own attempts to prove results and those slick classroom and textbook derivations they see every day. One should not neglect the rare opportunity this play offers to expose new role models for *both* teacher *and* student.

Lakatos' play has become a classic in its time, an argument for the heuristic method of teaching of Pólya, and a superb reconstruction of history. There are other mathematical dialogues to consider presenting as a follow-up: from Plato's *Dialogues*, through Galileo's *Dialogues on*



We must somehow treat counterexamples with more respect, and not stubbornly exorcise them by calling them monsters.

Two New Sciences, to John Pottage's recent *Geometrical Investigations*, but the cast in each of these is very small. The most recent mathematical play, suitable for high school audiences, is Ian D. Macdonald's *Alex*. More promising but much more demanding might be Donald Knuth's *Surreal Numbers*. Perhaps the best choice would be some of the mathematical dialogues in Douglas Hofstadter's *Gödel*, *Escher*, *Bach*. Requiring no props save the occasional Escher print, these dialogues are short, lively and independent.

The authors thank their friend and artist Andrew Dobrowolski.

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Experimental Gambling System

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You are seated in a large lecture hall listening to Professor Duval discuss the four points shown on the overhead projector. You responded to his call for volunteers to participate in an

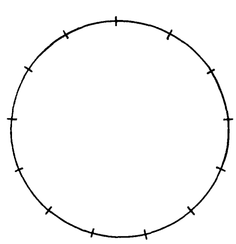
RISK ATTRACTIVES VS. RISK AVERTERS

"EXPERIMENTAL" GAMBLING SYSTEM. IDEAL PROPERTIES:

1. SIMPLICITY. No complex rules or gadgetry, as in casino games.
2. MEASURABILITY. Variable payoffs help quantify "threshold" behavior.
3. IMPARTIALITY. Risk presented in an unbiased, distraction-free environment.
4. REALISM. Non-use of "play money" reveals true attitudes.

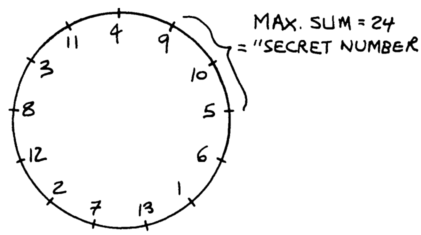
experiment, part of his research in human risk preference-avoidance. You wish he would get to the heart of the matter: the "substantial sum of money" promised to one of the volunteers.

Upon entering you received a card printed with the following configuration.



After emphasizing point (3), Duval instructs the audience to begin the experiment by writing the numbers 1 to 13 arbitrarily around the circle, one number in each of the thirteen positions. This is to be done privately and according to your own whim—the more random the better. Duval says that those who have "disobeyed" and written 1 to 13 in the natural order will be among those choosing to withdraw from the game once the rest of the rules are announced.

The next step is to compute the *largest sum given by three consecutive numbers* in the circular arrangement. This sum is a player's *secret number*. It is confidential and unalterable. Accuracy is a must here, since the winner will evidently have to authenticate his or her secret number. The audience gets busy computing and double-checking their secret numbers. You have written



Your secret number is 24. Duval instructs the audience to turn their cards over and not touch them any more.

Consistent with point (1), the game is simple. Players will bid an increasing sequence of integers, as in an auction, and one possibility is that the last bidder will win because higher bids are not forthcoming. If this is the case all other players *must* pay the winner $25 each. At this point, part of the audience begins gathering their things to depart. Duval appears amused at the predictable response and returns to point (4), arguing that this is probably the most important feature. Past experiments using play-money gave completely unrealistic pictures of behavior in "monetized risk environments." This is what social scientists must call gambling nowadays. In any event, today's experiment is totally voluntary, and those who are leaving are, in fact, participating in it: their numbers are carefully recorded by Duval's graduate assistant.

You will stay because not all of the rules of the game have been explained and there is still time to get out. So far it sounds exactly like an auction. Now the twist: there is another way to win, a more interesting way. Suppose someone bids a number which strictly exceeds your secret number. If that happens, you should signal it, quickly and emphatically, and become the winner! If you can underbid the current bidder using the value of your secret number and are the first to announce that you can, *you* become the last bidder—and the winner. The bidder before you, the one who was underbid, must then personally pay you $250. Everyone else goes home neither the richer nor poorer. Duval summarizes on the overhead:

```
        TODAY'S PAYOFFS:  $25, $250

PURSUE RISK (by bidding)-

At best: win $250, or else $25(m-1), m players
At worst: lose $250

AVOID RISK (by not bidding)-

At best: lose nothing
At worst: lose $25

STRATEGY: Raise the bid in unit increments.
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Someone asks if you can bid *any* number you want. Answer: Yes, just as long as it's higher than the previous bid. Why the strategy of raises in unit increments? Answer: It gives the smallest possible gap for an "underbidder" to jump into. Is this for real? Must a person who lost because he or she is underbid really pay $250 to the winner? Answer: Absolutely! In cash or with a credit card, but no checks, please. Now the audience is bustling with genuine panic. Many are moving toward the exits. About ten people remain, including yourself. It looks as if someone is going to win $250 one way or another. Duval says to let the game begin.

You hear the first bid: "One!," and everybody laughs. Every card has a 13 on it and next to that at least 1 and 2. So a secret number can't be less than 16. You bid 16. There must be quite a few "risk averters" here today because no one is responding. Duval cautions that unless someone raises the bid you are the winner as last bidder.

It begins to dawn on you that these bids are too low. What does the *average* consecutive-triple sum look like? If the numbers $1,2,\dots,n$ are written in circular permutation, to form the arithmetic mean of the triple sums you add up n such sums and divide by n. But that's the same as adding $1+2+\cdots+n$ to itself three times and dividing by n. Remembering the useful formula $1+2+\cdots+n=n(n+1)/2$ gets you the quick result: The mean of the consecutive-triple sums in any circular permutation is $3(n+1)/2$, so nobody is holding a secret number less than this. It's the old rule about the maximum being no less than the average. With $n=13$ this equals....

"Twenty-one!" You heard it just as you wanted to call it. While daydreaming about arithmetic means, the bidding went up. People are now getting the hang of it. The question is: can anyone actually be holding 21 as their secret number? In that case, don't dare bid 22. But if someone is holding 21, it is because all of their triple sums are 21 and the maximum *equals* the average. But that's impossible! If two adjacent triple sums are the same, you think $S_1=x_1+x_2+x_3=x_2+x_3+x_4=S_2$ and get $x_1=x_4$, an obvious contradiction in a situation with distinct integers $1,\dots,13$. No one is going to beat your bid of 22.

"Twenty-two!" said a female voice behind you. Now you understand why Duval called it a game of nerve and reflexes. You're putting too much thought into this. Your secret number is 24 and the current bid is 22. On the other hand, the worst possible secret number is $11+12+13=36$. That seems far from both 22 and 24. Probably lots of people are holding numbers like $27, 28, \dots$

"Twenty-three!" going once, going twice... Duval is getting ready to give your $25 to a young man in the first row. He's the one who bid 23 and no one else is responding with a higher bid. Now's the time:

"Twenty-four!" you call, and grit your teeth. Silence.

"Twenty-three! I win!" shouts the female voice. You have been underbid.

1. Swindle

Back in the office you're thinking about your new credit-card balance and the female undergraduate who is holding a claim on it. Perhaps it already occurred to you that you were swindled. Duval, an elegant scam artist, and the "female undergraduate" (his accomplice) did, in fact, take your money. It was nothing personal, just that as an amateur mathematician he's been challenged to create a few mathematical swindles to go along with his more lucrative "ventures." He explained how it works this way:

To begin with, the business about measuring risk preference-avoidance is pure distraction. The accomplice knows how to write the numbers $1,2,\dots,13$ in a circular permutation that gives the minimum possible secret number 23. The proof that 23 is the minimum is simple. The number 1 must go somewhere on the circle, so think of the remaining 12 positions partitioned into 4 disjoint triple sums. If these are 22 or less, we get that the sum of the numbers 1 to 13 is 89 or less, an unfortunate conclusion since $1+2+\cdots+13=91$. We leave it to the reader to confirm that 23 is an attainable secret number.

So the accomplice cannot be underbid. One way the swindle could fail is if someone else gets to 23 first and then the bidding stops. However, the accomplice, virtually with complete safety, can bid 24 herself! Duval said that this was the whole point, but that it was essentially a moot one. An effective MC will remind the audience that the last bidder wins. Someone else is bound to raise the bid above 23.

Now the critical complication: It is possible that another player has randomly hit 23 as *his* secret number. It's a question of the probability of such an event, and here is where the mathematics gets interesting. Although the possible secret numbers range from 23 to 36—14 choices—this probability is nothing like 1/14.

To get some feeling for it, write the numbers $1, 2, \ldots, 6$ randomly in a circle and compute the largest sum given by three consecutive terms. Notice that your result is 12 or larger, unless the following small miracle has occurred. With 6 placed first, 5 must be placed so: 6 _ _ 5 _ _ in order to preserve the true minimum, which is 11. Now where can 4 go? If 4 goes here 6 _ 4 5 _ _ then 1 comes next 6 1 4 5 _ _ followed by 6 1 4 5 2 3. On the other hand, if 4 goes there 6 _ _ 5 4 _, then we get 6 3 2 5 4 1, which is exactly the same circular permutation read counterclockwise. In fact, if we call two circular permutations the same if they agree under rotation and clockwise *or* counterclockwise reading, then the secret number minimization problem has a unique solution when $n = 6$. The number of all such circular permutations of $1, 2, \ldots, n$ is $(n-1)!/2$. So the probability of having randomly obtained 11 as your secret number when $n = 6$ is $1/60 = .0166\ldots$

If you tried to arrange the numbers $1, 2, \ldots, 13$ to obtain a secret number of 23, you already have some feeling for the probabilities.

Now consider a general version of the game, with $1, 2, \ldots, n$ arranged in a circle. We will show that the probability of hitting the minimum secret number goes to zero at a geometric rate as $n \to \infty$. Along the way there will be some interesting problems to solve.

2. Generalized secret numbers

Let σ denote a circular permutation of the numbers $1, 2, \ldots, n$; $\omega_{k,n}$ the largest sum given by k consecutive terms in σ; and $\overline{\omega}_{k,n} = \min_{\sigma} \omega_{k,n}$. We call sums of k consecutive elements in a circular permutation *k-sums*. When n is, say, 10,000, it's hard to find an arrangement which minimizes the largest 3-sum but not at all hard to find one which minimizes the largest 2-sum. That gives a clue to how to do it when k is even, which in turn provides clues to the case k odd. The problem is: What is the approximate size of $\overline{\omega}_{k,n}$ as $n \to \infty$?

LEMMA 1.

For fixed $k \geqslant 2$ *the sequence* $\{\overline{\omega}_{k,n}\}_{n=k}^{\infty}$ *is monotone nondecreasing.* (1)

For any σ *the arithmetic mean of the k-sums equals* $k(n+1)/2 < \overline{\omega}_{k,n}$. (2)

$$\text{When } k \text{ is even, } \overline{\omega}_{k,\lambda k} = \frac{\lambda k^2 + k + 2}{2}, \qquad \lambda = 2, 3, \ldots. \tag{3}$$

$$\text{When } k \text{ is odd } \frac{2\lambda k^2 + k + 1}{2} \leqslant \overline{\omega}_{k,2\lambda k} \leqslant \frac{2\lambda k^2 + k + 5}{2}, \qquad \lambda = 2, 3, \ldots. \tag{4}$$

$$\text{Therefore, } \lim_{n \to \infty} \overline{\omega}_{k,n}/n = k/2. \tag{5}$$

Proof. The proof of (1) is by contradiction. If $\{\overline{\omega}_{k,n}\}_{n=k}^{\infty}$ is not monotone nondecreasing, there must be some first n for which $\overline{\omega}_{k,n+1} < \overline{\omega}_{k,n}$. Choose σ^* so that $\omega_{k,n+1}(\sigma^*) = \overline{\omega}_{k,n+1}$. If we remove the number $n+1$ from σ^* and consider the remaining terms as a circular permutation σ' of $1, 2, \ldots, n$, the k-sums either stay the same or actually decrease—some of them are now missing what was the largest term in σ^*. But that gives an absurd result:

$$\overline{\omega}_{k,n} \leqslant \omega_{k,n}(\sigma') \leqslant \overline{\omega}_{k,n+1} < \overline{\omega}_{k,n}.$$

The proof of (2) uses an argument we saw before. Regardless of how the numbers $1, 2, \ldots, n$ are arranged, computing the arithmetic mean of the k-sums involves summing $1 + 2 + \cdots + n$, k times, and dividing by n. No $\omega_{k,n}$ can be less than $k(n+1)/2$ by the rule that the maximum cannot be less than the average, so neither can $\overline{\omega}_{k,n}$.

Consider (3). We construct a σ for which $\omega_{k,\lambda k}$ attains its minimum. This is done by arranging λ blocks, each of length k, side by side in σ. The typical block is

$$\begin{array}{ll} 0\lambda+j+1 & \\ 2\lambda-j & \\ 2\lambda+j+1 & \\ 4\lambda-j & \\ 4\lambda+j+1 & \\ 6\lambda-j & j=0,1,\ldots,\lambda-1. \\ 6\lambda+j+1 & \\ \vdots & \\ (k-2)\lambda+j+1 & \\ k\lambda-j & \end{array} \tag{6}$$

Setting $j=0$, we write these terms consecutively. Then we set $j=1$ and lay in the second block, and so on. For any j the sum of the elements in (6) is $(\lambda k^2/2)+(k/2)$, the average of the k-sums. It isn't possible for all the k-sums to equal the average, but it is possible for them to oscillate between the average and the average-plus-one as k-sums move stepwise around the circle. This is what happens in (6) except for some k-sums straddling the "wraparound point" in σ. These go below the average. Hence, the largest k-sum in σ is the average-plus-one: $(\lambda k^2+k+2)/2$. System (6) is a natural generalization of the way one would minimize the largest 2-sum by arranging pairs of the form $(j+1,2\lambda-j)$, $j=0,1,\ldots,\lambda-1$, when $n=2\lambda$.

The case k odd is trickier. A two-stage system is used in forming $\hat{\sigma}$ a circular permutation for which $\omega_{k,2\lambda k}(\hat{\sigma})$ equals the upper bound in (4). Again, the average dictates the lower bound

$$\frac{2\lambda k^2+k+1}{2}=\left\lceil\frac{k(2\lambda k+1)}{2}\right\rceil\leqslant\bar{\omega}_{k,2\lambda k}\leqslant\omega_{k,2\lambda k}(\hat{\sigma}),$$

where $\lceil x\rceil$ denotes the least integer not less than x. First we arrange blocks of type "A", below, side by side as j runs from 0 to $\lambda-1$. This is followed by blocks of type "B".

$$\left.\begin{array}{r} \lambda+j+1 \\ 3\lambda-j \\ 5\lambda+j+1 \\ 7\lambda-j \\ 9\lambda+j+1 \\ 11\lambda-j \\ \vdots \\ 2\lambda k-7\lambda-j \\ 2\lambda k-6\lambda+j+1 \\ 2\lambda k-3\lambda+j+1 \\ 2\lambda k-2j \end{array}\right\} j=0,1,\ldots,\lambda-1 \qquad \left.\begin{array}{r} 0\lambda+j+1 \\ 4\lambda-j \\ 4\lambda+j+1 \\ 8\lambda-j \\ 8\lambda+j+1 \\ 12\lambda-j \\ \vdots \\ 2\lambda k-6\lambda-j \\ 2\lambda k-5\lambda+j+1 \\ 2\lambda k-4\lambda+j+1 \\ 2\lambda k-2j-1 \end{array}\right\} j=0,1,\ldots,\lambda-1.$$

Type "A" Type "B"

For example, when $k=3$, only the last three terms in "A" are used in a block and likewise for "B." When $k=5$, it's the first two terms and the last three terms, and so on. It's true that there is some loss of precision in going from (3) to (4), but for a different block system which hits minimum secret numbers exactly, see the concluding Notes.

We prove (5) for the case k even (analogously for k odd). Let n be arbitrary and choose λ so that $(\lambda-1)k<n\leqslant\lambda k$. By (1), (2), and (3)

$$\frac{k(n+1)}{2n}\leqslant\frac{\bar{\omega}_{k,n}}{n}\leqslant\frac{\lambda}{\lambda-1}\left(\frac{\lambda k^2+k+2}{2\lambda k}\right). \tag{7}$$

Taking limits across (7) as $\lambda,n\to\infty$ gives (5).

3. Kaplansky's lemmas

When I asked "Duval" about his alias he explained that he used it in honor of the French classical school of combinatorics, mathematicians like Bertrand, André, and François Eduard Anatole Lucas, inventor of the famous *problème des ménages*. This problem asks for the number of ways of seating m husbands and their m wives around a circular table, men and women alternating, so that no husband sits next to his wife. In a sense, it is a distant cousin to the problem we're considering here. In order to get the smallest secret number in a circular permutation there *are* objects which must be kept apart: the "heavyweights" in $1,2,\ldots,n$, numbers like $n, n-1,\ldots$

Although Lucas lived in the 19th century, the *problème des ménages* continued to build a considerable literature well into the 20th century. It was solved brilliantly by Irving Kaplansky in 1943 [**1**]. In his solution, Kaplansky proved two lemmas having to do with the number of ways of selecting m objects from n objects, first arrayed linearly and then in a circle, subject to a separation condition: no two selected objects could be adjacent.

In this section we generalize Kaplansky's lemmas. In arranging the numbers $1,2,\ldots,13$ in a circle it's not good enough for 12 and 13 to be "not adjacent." If they are separated by just one other number, they will contribute to a triple sum which is at least 26. Remember that the winning secret number is 23.

In the following lemma we enumerate a "difficult" set by establishing a one-to-one correspondence with another set whose elements are much easier to count.

LEMMA 2. *The number of ways of selecting m objects from n objects arrayed in a row, with every pair of selected objects separated by at least r unselected objects, is* $\binom{n-mr+r}{m}$.

Proof. Consider the problem of counting selections of m volumes, no two consecutive, from the n numbered volumes of an encyclopedia ($r=1$, the case proved in [**1**]). Imagine withdrawing, say, 5 volumes from a 22-volume set, then compressing the remaining volumes together. How would these actions appear if run in reverse? You would see 5 uniquely defined spaces opening up between the remaining 17 volumes and your selection moving back into the correct spaces. This is a one-to-one correspondence. Two volumes replaced in two adjacent spaces are still separated by one volume. But enumerating the set of unrestricted selections of 5 "cracks" between 17 volumes is easy: the number of such selections is $\binom{18}{5}$ (the extra crack is at the book ends). Obviously we could have used m in place of 5 and n in place of 22. This completes the first step in a proof by induction on r.

Now let $r>1$ be arbitrary. Repeating the experiment we find that we are unable to replace volumes in adjacent cracks. Pairs of selected cracks must have at least $r-1$ cracks between them. But this is the same problem, ready for the induction hypothesis. We have $n-m+1$ objects (cracks) arrayed in a row from which we must choose m with at least $r-1$ separating each chosen pair. The number is

$$\binom{n-m+1-(r-1)m+(r-1)}{m}=\binom{n-mr+r}{m}.$$

LEMMA 3. *The number of ways of selecting m objects from n objects arrayed in a circle, with every pair of selected objects separated (along both arcs) by at least r unselected objects, is* $\frac{n}{n-mr}\binom{n-mr}{m}$.

Proof. Think of the objects as points on a circle. Call a subset of r consecutive *unselected* points an *r-gap*. For every r-gap there are $n-r$ remaining points from which we select m, subject to the separation constraint of Lemma 2. There are $\binom{(n-r)-mr+r}{m}=\binom{n-mr}{m}$ such selections. Now imagine an r-gap moving stepwise around the circle. As it does we enumerate, with redundancy, $n\binom{n-mr}{m}$ selections of m points. We want to factor out the overcounting.

Let S denote the set of sought-for selections and $|S|$ the number of elements in S. We have just counted each element s in S as many times as there are r-gaps which contain no point of s. But there are always $n-mr$ of these. So

$$(n-mr)|S| = n\binom{n-mr}{m}.$$

Now arrange the numbers $1,2,\ldots,n$ randomly in a circle and compute the largest sum of three consecutive terms. How likely is it that you've hit the winning secret number, the smallest possible value among all circular permutations?

THEOREM. *Let E_n denote the event $\{\sigma\colon \omega_{3,n}(\sigma) = \bar{\omega}_{3,n}\}$ and $P(E_n)$ the probability of E_n. As n becomes large*

$$P(E_n) < \left(1 - \frac{2\alpha}{n-1}\right)^{\alpha-1},$$

where α is the greatest integer not greater than $n/6$. Hence, $\lim_{n\to\infty} P(E_n) = 0$ with at least geometric rate of convergence.

Proof. Choose a convenient value greater than $3/2$ but less than 2, like $5/3$. Observe that as a player places $1,2,\ldots,n$ around the circle he chooses, in the process, random positions for the largest of the numbers, namely, for $S_n = \{n, n-1,\ldots, n-\alpha+1\}$. If any two elements of S_n should fall adjacent or with just one other number between them, they will contribute to a triple sum which strictly exceeds $5n/3$. But Lemma 1-(5) says that for sufficiently large n the minimum secret number is strictly less than $5n/3$. Therefore, a necessary condition for E_n to occur is that another event, A_n, occur. A_n is the event that every pair of positions selected for members of S_n is separated by at least two positions not assigned to members of S_n. $P(E_n)$ is dominated by $P(A_n)$.

By Lemma 3, with $r=2$,

$$P(A_n) = \frac{\frac{n}{n-2\alpha}\binom{n-2\alpha}{\alpha}}{\binom{n}{\alpha}} < \left(1 - \frac{2\alpha}{n-1}\right)^{\alpha-1}.$$

But $(1 - \frac{2\alpha}{n-1})^{\alpha-1}$ is asymptotically like $(2/3)^{n/6}$. $P(A_n)$, hence $P(E_n)$, is being driven to zero geometrically fast as $n \to \infty$.

4. Notes and acknowledgment

An anonymous referee suggested a beautiful method for producing circular permutations which *attain* $\bar{\omega}_{3,6\lambda} = 9\lambda + 3$, $\lambda = 2,3,\ldots$ First put down blocks of the form $(6\lambda - 2j, 2\lambda + j + 1, \lambda + j + 2)$, $j = 0,1,\ldots,\lambda - 2$, followed by the single block $(4\lambda + 2, 3\lambda, \lambda + 1)$, followed by blocks of the form $(4\lambda + 2j + 1, 4\lambda - j, \lambda - j)$, $j = 0,1,\ldots,\lambda - 1$. For example, when $n = 6$ this method yields, as it must, 6 3 2 5 4 1. When $n = 12$, we get 12 5 4 10 6 3 9 8 2 11 7 1 and an unbeatable secret number of 21. There are many other arrangements which work, like 12 1 8 11 2 7 10 3 6 9 4 5.

The authors wish to thank Professor James Lewis for helpful discussions, and the two referees for important contributions to this paper.

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The Harmonic Triangle and the Beta Function

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The Harmonic Triangle

$$\begin{array}{cccccc}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdot \\
 & \frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \cdot \\
 & & \frac{1}{3} & \frac{1}{12} & \frac{1}{30} & \cdot \\
 & & & \frac{1}{4} & \frac{1}{20} & \cdot \\
 & & & & \frac{1}{5} & \cdot \\
 & & & & & \cdot
\end{array}$$

is the difference table of the harmonic sequence. Although this triangle is not as widely known as Pascal's triangle, it has been around for quite a while. In fact, its origin can be traced back to the seventeenth century when Leibniz used the sequential differences in summing some infinite series [**1**], [**2**], and [**4**]. For this reason the harmonic triangle is sometimes called Leibniz' triangle. Here, we will show that the entries of this array are the values of the beta function

$$\beta(\mu, \nu) = \int_0^1 x^{\mu-1}(1-x)^{\nu-1}\,dx$$

for the integral parameters μ, ν with $\mu \geq 1$, $\nu \geq 1$.

Using this fact, we will be able to justify the symmetry within the harmonic triangle and to find row sums and column sums in this triangle. Also, we will derive the basic relationship between the harmonic triangle and Pascal's triangle.

If we number the columns and rows starting with zero, then the entry in the nth column and rth row—which we denote by H_r^n—is the difference $\Delta^r u_{n-r}$. A "well-known" difference formula [**3**] tell us that:

$$\Delta^r u_{n-r} = u_{n-r} - \binom{r}{1} u_{n-r+1} + \binom{r}{2} u_{n-r+2} - \cdots + (-1)^r u_n,$$

where the u_n's are terms in the zero row. Hence,

$$H_r^n = \frac{1}{n-r+1} - \binom{r}{1}\frac{1}{n-r+2} + \binom{r}{2}\frac{1}{n-r+3} - \cdots + (-1)^r \frac{1}{n+1}.$$

But

$$\int_0^1 x^{n-r}(1-x)^r\,dx = \int_0^1 \left[x^{n-r} - \binom{r}{1}x^{n-r+1} + \binom{r}{2}x^{n-r+2} - \cdots + (-1)^r x^n\right] dx$$

$$= \frac{1}{n-r+1} - \binom{r}{1}\frac{1}{n-r+2} + \binom{r}{2}\frac{1}{n-r+3} - \cdots + (-1)^r \frac{1}{n+1}.$$

Thus, $H_r^n = \int_0^1 x^{n-r}(1-x)^r\,dx$ for all integers $n \geqslant r \geqslant 0$. And by the definition of the beta function, we have

$$H_r^n = \beta(n-r+1, r+1) \quad \text{for all integers } n \geqslant r \geqslant 0.$$

Now we may conclude the following:

(1) The symmetry within each column, namely $H_r^n = H_{n-r}^n$ is due to the equality

$$\beta(n-r+1, r+1) = \beta(r+1, n-r+1),$$

each side being equal to $\Gamma(n-r+1)\Gamma(r+1)/\Gamma(n+2)$.

(2) If $r \geqslant 1$, then the sum of the entries in the rth row is

$$\sum_{k=r}^{\infty} H_r^k = \sum_{k=r}^{\infty} \int_0^1 x^{k-r}(1-x)^r \, dr = \int_0^1 \frac{1}{1-x}(1-x)^r \, dx$$
$$= \int_0^1 (1-x)^{r-1} \, dx.$$

Thus,

$$\sum_{k=r}^{\infty} H_r^k = H_{r-1}^{r-1} \quad \text{for all } r \geqslant 1.$$

As an example, the sum of the entries in the second row is

$$\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \cdots = \frac{1}{2}.$$

(3) To find the sum of the entries in the nth column, we have

$$\sum_{r=0}^{n} H_r^n = \sum_{r=0}^{n} \int_0^1 x^{n-r}(1-x)^r \, dx = \int_0^1 \frac{x^{n+1} - (1-x)^{n+1}}{x-(1-x)} \, dx.$$

Using the transformation $x = \frac{1}{2} + y$, we get

$$\sum_{r=0}^{n} H_r^n = \left(\frac{1}{2}\right)^n \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2r+1} / (2r+1).$$

(4) Since

$$H_r^n = \beta(n-r+1, r+1) = \frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(n+2)} = \frac{(n-r)!r!}{(n+1)!},$$

then

$$H_r^n = \frac{1}{(n+1)\binom{n}{r}}.$$

This equality gives us the relationship between the harmonic triangle and Pascal's triangle

1	1	1	1	1	•
	1	2	3	4	•
		1	3	6	•
			1	4	•
				1	•

(5) If we use the previous relation together with the column sum in (3) we get the identity

$$\frac{1}{n+1} \sum_{r=0}^{n} \binom{n}{r}^{-1} = \left(\frac{1}{2}\right)^n \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2r+1} / (2r+1),$$

or, equivalently,

$$\sum_{r=0}^{n} \binom{n}{r} \cdot \sum_{r=0}^{n} \binom{n}{r}^{-1} = (n+1) \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2r+1} / (2r+1).$$

Also, the relation in (4) implies that $\sum_{r=0}^{n}\binom{n}{r}H_r^n = 1$.

References

[1] M. Bicknell-Johnson, Diagonal sums in the harmonic triangle, The Fibonacci Quarterly, 19 (1981) 196–198.
[2] C. B. Boyer, A History of Mathematics, John Wiley & Sons, Inc, N.Y., 1968, p. 439–440.
[3] G. Chrystal, Algebra, Part II, Chelsea Publishing, N.Y., 1964, p. 401.
[4] G. Pólya, Mathematical Discovery, John Wiley & Sons, Inc., N.Y., 1981, p. 89.

A Note on Evaluating Limits Using Riemann Sums

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In this note, we present a method for evaluating some limits. This method, which does not appear to be widely used, involves expressing the limit in the form of a Riemann sum. This sum, in turn, is equivalent to a definite integral, which can be evaluated using standard techniques. We illustrate the technique with several examples.

The first example appears as an exercise in the book *Advanced Calculus* by David V. Widder [**2**, p. 391]. It can also be solved using Stirling's formula or series methods.

Example 1. Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right).$$

Solution. We have

$$\begin{aligned}\lim_{n\to\infty}&\left(\frac{1}{n}\ln n! - \ln n\right)\\ &= \lim_{n\to\infty}\left[\frac{1}{n}\left(\sum_{k=1}^{n}\ln k\right) - \ln n\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{k=1}^{n}(\ln k - \ln n)\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right).\end{aligned}$$

This limit is the Riemann sum of $f(x) = \ln x$ over the interval $[0,1]$. Hence,

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right) = \int_0^1 \ln x\, dx.$$

Integrating this improper integral by parts, and using l'Hôpital's rule, we get

$$\int_0^1 \ln x\, dx = -1.$$

Also, the relation in (4) implies that $\sum_{r=0}^{n}\binom{n}{r}H_r^n = 1$.

References

[1] M. Bicknell-Johnson, Diagonal sums in the harmonic triangle, The Fibonacci Quarterly, 19 (1981) 196–198.
[2] C. B. Boyer, A History of Mathematics, John Wiley & Sons, Inc, N.Y., 1968, p. 439–440.
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$$\int_0^1 \ln x\,dx = -1.$$

Hence,

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right) = -1.$$

Our next example is a problem which was proposed by Norman Schaumberger in the March 1984 issue of *The College Mathematics Journal* [3].

Example 2. Evaluate

$$\lim_{n\to\infty}\frac{1}{n}[2\cdot 5\cdot 8\cdots(3n-1)]^{1/n}.$$

Solution. Let

$$y=\frac{1}{n}[2\cdot 5\cdot 8\cdots(3n-1)]^{1/n}.$$

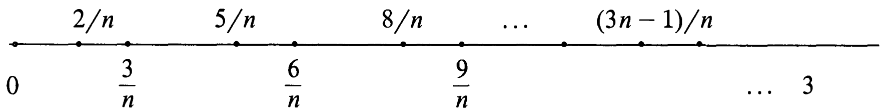
Taking ln of both sides, we obtain

$$\begin{aligned}\ln y &= \frac{1}{n}\left[\sum_{k=1}^{n}\ln(3k-1)\right] - \ln n\\ &= \frac{1}{n}\sum_{k=1}^{n}[\ln(3k-1) - \ln n]\\ &= \frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{3k-1}{n}\right).\end{aligned}$$

Taking the limit as $n\to\infty$ of both sides, we obtain

$$\begin{aligned}\lim_{n\to\infty}\ln y &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{3k-1}{n}\right)\\ &= \frac{1}{3}\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\ln\left(\frac{3k-1}{n}\right).\end{aligned} \qquad \text{(A)}$$

The last expression in (A) above is a Riemann sum for the function $f(x)=\ln x$ over the interval $[0,3]$. This is clear with the help of the following figure.



Thus,

$$\ln\left(\lim_{n\to\infty} y\right) = \frac{1}{3}\int_0^3 \ln x\,dx$$

Integrating this improper integral by parts, and using l'Hôpital's rule, we get

$$\ln\left(\lim_{n\to\infty} y\right) = \frac{1}{3}(3\ln 3 - 3) = \ln 3 - 1.$$

Thus,

$$\lim_{n\to\infty} y = e^{(\ln 3 - 1)} = \frac{3}{e}.$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n}[2\cdot 5\cdot 8\cdots(3n-1)]^{1/n} = \frac{3}{e}.$$

Note. Example 2 above can be generalized as follows:

$$\lim_{n\to\infty} \frac{1}{n}[(k-1)(2k-1)(3k-1)\cdots(kn-1)]^{1/n} = \frac{k}{e} \quad \text{for } k=2,3,4,\dots.$$

We tried to prove Stirling's formula using the above technique but we were not successful. However, we were able to prove the following result which is very similar to Stirling's formula.

Example 3.

$$\lim_{n\to\infty}\left(\frac{n!e^n}{n^{n+(1/2)}\sqrt{2\pi}}\right)^{1/n} = 1.$$

Solution. Let

$$y = \left(\frac{n!e^n}{n^{n+(1/2)}\sqrt{2\pi}}\right)^{1/n}.$$

Taking the natural log of both sides, we obtain

$$\begin{aligned}\ln y &= \frac{1}{n}\left[\ln n! + n\ln e - \left(n+\frac{1}{2}\right)\ln n - \frac{1}{2}\ln 2\pi\right]\\ &= \frac{1}{n}\ln n! + 1 - \ln n - \frac{\ln n}{2n} - \frac{\ln 2\pi}{2n}\end{aligned}$$

Taking the limit as $n\to\infty$ of both sides, we obtain

$$\begin{aligned}\lim_{n\to\infty}\ln y &= \lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right) + 1\\ &= -1+1 \text{ (using example 1)}\\ &= 0.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} y = e^0 = 1.$$

We now illustrate how this technique can be used in approximating partial sums of divergent p-series.

Example 4. We know that the p-series $\sum_{k=1}^{\infty} k^p$ converges if $p < -1$ and diverges otherwise. In this example, we show that for large n and $p > -1$, $\sum_{k=1}^{n} k^p$ is approximately $n^{p+1}/(p+1)$ in the sense that the limit of the ratio of the two quantities as $n\to\infty$ is 1. We have

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^p}{n^{p+1}} &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p\\ &= \int_0^1 x^p\,dx\\ &= \left(\frac{x^{p+1}}{p+1}\right)\Bigg|_0^1\\ &= \frac{1}{p+1} \quad \text{for } p > -1.\end{aligned}$$

Hence for large n and $p > -1$, we have $\sum_{k=1}^{n} k^p$ is approximately $n^{p+1}/(p+1)$. We observe that $n^{p+1}/(p+1)$ is the first term in the approximation of p-series by Riemann-Zeta functions [4]; however, this requires sophisticated machinery in comparison to the method described above.

With the aid of a computer, we calculated the value of $\sum_{k=1}^{n} k^p$ and $n^{p+1}/(p+1)$ for various values of n and p. Our results (summarized in TABLES 1 and 2 below) show that the percent error is small.

TABLE 1

n	p	$\sum_{k=1}^{n} k^p$	$\frac{n^{p+1}}{p+1}$	% Error
100	1	5050	5000	.99
100	2	338350	333333.33	1.48
100	3	25502500	25000000	1.97
100	4	2050333330	2000000000	2.45
200	1	20100	20000	.50
200	2	2686700	2666666.67	.75
200	3	404010000	400000000	.99
200	4	64802666660	64000000000	1.24
400	1	80200	80000	.25
400	2	21413400	21333333.33	.37
400	3	6432040000	6400000000	.49
400	4	2060821333320	2048000000000	.62

Notice that for fixed p, the % error decreases as n increases. Computer results for large values of n and p are given in TABLE 2 below.

TABLE 2

n	p	% Error
300	14	2.46
500	13	1.39
700	12	.92
800	12	.81
1000	11	.60

There exist many other limits that can be evaluated using Riemann sums, three of which are listed below.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}. \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=n+1}^{rn} \frac{1}{k} = \ln r \quad \text{for } r = 2,3,4,\dots. \tag{2}$$

$$\lim_{n\to\infty} \sum_{k=np}^{nq} \frac{1}{k} = \ln\left(\frac{q}{p}\right) \text{ for positive integers } p \text{ and } q \text{ with } p < q. \tag{3}$$

Limit (1) above appears in most advanced calculus texts and is usually solved either by Stirling's formula or by series methods. Limits (2) and (3) above appear as exercises in Widder's Advanced Calculus Book [**2**, p. 392] in the section dealing with Stirling's formula.

References

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Packing a Square Lattice with a Rectangle-Free Set of Points

Restating a problem in a different environment could be a key to a solution

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This paper originated when I was perusing the book *Mathematical Gems* III [**2**] by Ross Honsberger. Honsberger has an engaging way of writing, but even better than this, he has excellent taste in his choice of problems. I have had a long-term interest in the construction of problems for undergraduate and high school mathematical contests. One of the techniques that I and many others have used is to take some topic of modern mathematics and to construct a sufficiently simple example so that a solution can be found which does not require the solver to have a knowledge of the theory on which the problem is based. As a corollary, when I look at problems composed by others, I have a dual interest, namely: first to solve the problem and second, to try and find out what led the composer to the problem.

With this in mind, I looked at two problems which appeared in *Mathematical Gems* III. These problems and their solutions appear on pages 2–6. They are both problems that appeared in the 1974 USSR National Olympiad (high school).

Problem 1. Consider a square grid S of 169 points which are uniformly arrayed in 13 rows and 13 columns (like the lattice points (m,n), $m,n=1,2,\dots,13$). Prove that no matter what subset T, consisting of 53 of these points, might be selected, some 4 points of T will be the vertices of a rectangle R whose sides are parallel to the sides of S.

Problem 2. Given a square grid S of 49 points in 7 rows and 7 columns, a subset T of k points is selected. The problem is to find the maximum value of k such that no 4 points of T determine a rectangle R having sides parallel to the sides of S for an appropriate choice of the points of T.

With respect to these problems I note the following. In the first, the question of whether 53 is a best value, i.e., whether, with 52 points, an appropriate rectangle avoiding selection exists is not asked. In the second, both questions are asked, i.e., an appropriate set of k points exists but no set of $k+1$ points is rectangle avoiding. In what follows, a possible explanation of why the distinction was made is given.

Restating the problems in a different environment

In general when a problem is restated in a new environment the solution in the new environment is of the same degree of difficulty as in the old. There are important exceptions, however. First, for reasons of visualization or other psychological matters, the solver may be more comfortable working in the new setting. Second, and of greater importance, the new setting may contain a large body of results that the solver may bring to bear on the problem.

For the problems at hand we will restate them in terms of the theory of matrices. The initial grid S of points will be replaced by a matrix whose entries consist entirely of 0's and 1's. In the places where the points of T appear the matrix entry is 1, and at other points the entries are 0.

Suppose now that the matrix contains a rectangle of 1's. Looking at the two rows that contain the rectangle of 1's, it is clear that the scalar product of these two rows $\geqslant 2$. Similarly, the two columns which contain the same rectangle of 1's have a scalar product $\geqslant 2$. On the other hand, if a pair of rows, or columns, does not have a rectangle of 1's, then the corresponding scalar product has a value of 0 or 1. The next thing to note is that if A is a matrix and A^T is its transpose, then the matrix AA^T has as its entry in the ith row and jth column the scalar product of the ith and jth rows of A. Similarly, the same result is true for A^TA where the scalar product is that of the ith and jth columns of A. For the main diagonal entries of AA^T, the entry in the ith row and ith column is the number of 1's in the ith row of A. The same result holds for A^TA with row and column interchanged. In this new setting the general form of problem 2 (i.e., 7 is replaced by n) becomes:

Problem 2*. Let A be a square matrix of order n whose entries are confined to the values 0 or 1. Find the maximum value of k such that A has exactly k 1's and such that for some choice of places at which the 1's appear, the matrices AA^T and A^TA both have only 0's and 1's in all positions off the main diagonal.

Problems such as this occur in the theory of finite projective planes. The techniques for handling these problems in the matrix setting have been developed by H. J. Ryser, and a very readable account appears in Chapter 8 of [3]. In the next section I give a brief account of the elementary theory of finite projective planes. This contains more than enough to solve the problem at hand. I justify this with my view that a good problem is one in which the solution whets the appetite for more mathematics.

Finite projective planes

Without being too formal, we define a projective plane of order n as a system with two sets of elements called points and lines, and a relation of incidence between the elements of the two sets. If the relation is satisfied for a point p and a line l, we say that the point p is on the line l, or alternatively, that the line l passes through the point p. The points and lines of the system satisfy the following axioms:

1) Any two distinct points are on exactly one line.
2) Any two distinct lines pass through exactly one point.
3) There is a set of four distinct points, no three of which are on the same line.
4) There is a line which contains exactly $n+1$ points.

From these axioms it is very easy to show that the following hold:

a) Every line contains exactly $n+1$ points.
b) Every point is on exactly $n+1$ lines.
c) The number of points in the plane is exactly n^2+n+1.
d) The number of lines in the plane is exactly n^2+n+1.

We now introduce the notion of an incidence matrix for a projective plane of order n. First the n^2+n+1 points (in any order) are indexed with the integers from 1 to n^2+n+1, the point with index i being denoted by p_i. Similarly, the lines are subscripted in the same way. We construct a square matrix A of order n^2+n+1 by making the entry in the ith row and jth column equal 1 if p_i is on l_j and equal 0 otherwise. It is now clear that the matrices AA^T and A^TA both have all the entries which are off the main diagonal equal to 1, and the main diagonal entries equal $n+1$. Also to be noted is that the number of 1's in A is $(n+1)(n^2+n+1)$, since each of the n^2+n+1 rows contains $(n+1)$ 1's. This is more than enough to prove the following theorem, which we state in the form of points in a square lattice.

THEOREM 1. *Let n be an integer for which a projective plane of order n exists. Let $N=n^2+n+1$. Let S be a square grid of N^2 points which are arrayed in N rows and N columns. Then there exists a set T of $k=(n+1)(n^2+n+1)$ points in the array such that no four of these determine a rectangle with sides parallel to the sides of S.*

Proof. From the paragraph preceding the statement of the theorem it is clear that the positions of the 1's in the incidence matrix of the projective plane serve as an appropriate set of k points on the grid S.

We remark that we have not proved that the only selections of points in the square grid are those arising from the places where 1's appear in the incidence matrix of some projective plane. Nevertheless, we can show that the value of k used in Theorem 1 is the best possible. In this theorem the existence of a projective plane of order n is not necessary. The argument used is that which appears in Honsberger's book for the first problem. This is a type of argument which has been used by Ryser in a number of his papers.

THEOREM 2. *Let n be any positive integer and let $N = n^2 + n + 1$. Let $k = (n+1)(n^2+n+1)$. Let S be a square grid of N^2 points which are arrayed in N rows and N columns. Let T be any set of $k+1$ nodes of the grid. Then T contains four nodes which are the vertices of a rectangle with sides parallel to the sides of S.*

Proof. Let the ith row of S contain a_i nodes of T for $i = 1, \dots, N$. The number of pairs of nodes of T which occur in the ith row of S is $a_i(a_i - 1)/2$. Each such pair determines a pair of column indices. The sum

$$m = a_1(a_1 - 1)/2 + a_2(a_2 - 1)/2 + \cdots + a_N(a_N - 1)/2$$

counts all the pairs of column indices of all the pairs of nodes of T which occur in the rows of S. The total number of distinct pairs of column indices for the whole array is $N(N-1)/2$. Hence, if we can prove that $m > N(N-1)/2$, then by the pigeonhole principle it would follow that a pair of column indices would be common to two rows and this would imply that T would contain four nodes at the corners of a rectangle. Thus we must prove that

$$a_1(a_1 - 1) + a_2(a_2 - 1) + \cdots + a_N(a_N - 1) > N(N-1).$$

We also note that

$$a_1 + a_2 + \cdots + a_N = k + 1 = (n+1)(n^2+n+1) + 1.$$

Using this equation the inequality becomes

$$a_1^2 + a_2^2 + \cdots + a_N^2 > N(N-1) + (n+1)(n^2+n+1) + 1.$$

Substituting for N we get

$$a_1^2 + a_2^2 + \cdots + a_N^2 > (n+1)^2(n^2+n+1) + 1.$$

We now establish this inequality. Using the Cauchy inequality

$$\left(a_1^2 + a_2^2 + \cdots + a_N^2\right)(1^2 + 1^2 + \cdots + 1^2) \geqslant (a_1 + a_2 + \cdots + a_N)^2$$

we obtain

$$\left(a_1^2 + a_2^2 + \cdots + a_N^2\right)(n^2+n+1) \geqslant \left((n+1)(n^2+n+1) + 1\right)^2.$$

This reduces to

$$\begin{aligned}\left(a_1^2 + a_2^2 + \cdots + a_N^2\right) &\geqslant (n+1)^2(n^2+n+1) + 2(n+1) + 1/(n^2+n+1) \\ &> (n+1)^2(n^2+n+1) + 1.\end{aligned}$$

We next look at a weakened form of Theorem 1 for the case where the assumption that there exists a projective plane of order n is not made. Again the argument used imitates that given in [**2**].

THEOREM 3. *Let $N = n^2 + n + 1$. Let S be a square grid of N^2 points which are arrayed in N rows and N columns. Let $k = (n+1)(n^2+n+1)$. If there is a set T of k nodes in the grid no four of which form the vertices of a square with sides parallel to the sides of S then each row and each column of S contains exactly $n+1$ nodes of T.*

Proof. Using the notation of the previous theorem, a necessary condition for the set T to exist is that

$$a_1(a_1-1)+a_2(a_2-1)+\cdots+a_N(a_N-1)\leqslant N(N-1).$$

Suppose T has k nodes where k is an integer not necessarily that given by the value in the statement of the theorem. Then

$$a_1+a_2+\cdots+a_N=k$$

and the inequality reduces to

$$a_1^2+a_2^2+\cdots+a_N^2\leqslant N(N-1)+k.$$

Substituting for N, the inequality becomes

$$a_1^2+a_2^2+\cdots+a_N^2\leqslant(n^2+n+1)(n^2+n)+k.$$

By the Cauchy inequality

$$(a_1^2+a_2^2+\cdots+a_N^2)(1^2+1^2+\cdots+1^2)\geqslant(a_1+a_2+\cdots+a_N)^2,$$

equality holding iff $a_1=a_2=\cdots=a_N$. Hence

$$a_1^2+a_2^2+\cdots+a_N^2\geqslant k^2/(n^2+n+1).$$

Combining the two inequalities we obtain

$$k^2/(n^2+n+1)\leqslant(n^2+n+1)(n^2+n)+k.$$

Hence,

$$k^2-(n^2+n+1)k-(n^2+n+1)^2(n^2+n)\leqslant 0,$$

or

$$(k+n^3+n^2+n)(k-(n^2+n+1)(n+1))\leqslant 0.$$

The largest value of k for which this inequality holds is $k=(n^2+n+1)(n+1)$, and this can only occur if

$$a_1=a_2=\cdots=a_N=n+1.$$

This establishes the theorem.

For the case where N is an integer not of the form n^2+n+1, one can obtain weakened forms of the theorems by using for n the positive root of the equation $x^2+x+1=N$, and then replacing n by nearby integers in the final statement of the theorem but we will not pursue this any further.

Construction of projective planes

There is a very extensive literature dealing with finite projective planes. For our purposes the existence of a single projective plane of order n is enough to determine an appropriate set of nodes T which satisfy the conditions of Theorem 1. Our present state of knowledge is that all known projective planes are of prime power order and that every prime power is the order of at least one projective plane. A good basic introduction to the construction of projective planes is given in the last six chapters of *Combinatorial Theory* by Marshall Hall, Jr. [**1**]. For small values of n an easy way to construct the plane is with the use of a difference set. We give a restricted definition of a difference set which leads to the construction of a projective plane. Let n be an integer and let $N=n^2+n+1$. Let $\{b_1,b_2,\ldots,b_{n+1}\}$ be $n+1$ integers such that the set

$$\{b_i-b_j|i,j=1,2,\ldots,n+1;\ i\neq j\}$$

contains all the nonzero residues (mod N). It is clear that if we add the same integer i to each of the elements of a difference set, the new set is also a difference set, which we call the ith translate of the set. From a difference set a projective plane of order n is obtained as follows: the $n+1$ elements of the difference set are the points of one line, and the remaining lines are obtained by taking the ith translates of this set for $i = 1, 2, \ldots, N-1 \pmod{N}$. For instance, for $n = 2$, $N = 7$, it is easily checked that $\{1,2,4\}$ is a difference set. The lines of the plane are the sets

$$\{1,2,4\}, \{2,3,5\}, \{3,4,6\}, \{4,5,7\}, \{5,6,1\}, \{6,7,2\}, \{7,1,3\}.$$

Note that the set $\{1,2,6\}$ could also have been used as a difference set, although the geometry obtained from this set is isomorphic to the original geometry.

For small values of n it is easy to find a difference set experimentally. To make things easier, one notes that since the translates of a difference set are all difference sets and since one of the differences is 1, we may start the difference set with $\{1,2,\ldots\}$, and continue by trial and error. For the case $n = 3$, $N = 13$ one easily finds that $\{1,2,4,10\}$ is a difference set. For $n = 4$, $N = 21$, a suitable set is $\{1,2,7,9,19\}$, or $\{1,2,5,15,17\}$.

Returning now to the original problem 1, we show how 53 is the best possible by exhibiting a 13×13 lattice with 52 points chosen so that no 4 are on a rectangle. The numbers in each row are translates of $\{1,2,4,10\}$.

1	2	4	10	×	×	●	×	●	●	●	●	●	×	●	●	●
2	3	5	11	●	×	×	●	×	●	●	●	●	●	×	●	●
3	4	6	12	●	●	×	×	●	×	●	●	●	●	●	×	●
4	5	7	13	●	●	●	×	×	●	×	●	●	●	●	●	×
5	6	8	1	×	●	●	●	×	×	●	×	●	●	●	●	●
6	7	9	2	●	×	●	●	●	×	×	●	×	●	●	●	●
7	8	10	3	●	●	×	●	●	●	×	×	●	×	●	●	●
8	9	11	4	●	●	●	×	●	●	●	×	×	●	×	●	●
9	10	12	5	●	●	●	●	×	●	●	●	×	×	●	×	●
10	11	13	6	●	●	●	●	●	×	●	●	●	×	×	●	×
11	12	1	7	×	●	●	●	●	●	×	●	●	●	×	×	●
12	13	2	8	●	×	●	●	●	●	●	×	●	●	●	×	×
13	1	3	9	×	●	×	●	●	●	●	●	×	●	●	●	×

Conclusion

It is almost inevitable that the composer of problems 1 and 2 started with the notion of an incidence matrix of a projective plane. However, the candidates for whom the examination was intended could not be expected to have such knowledge. In problem 2, the parameters are sufficiently small that the problem did not need this knowledge. In problem 1, the construction of an example such as that given in the previous section would require an intimate knowledge of finite plane theory. As a result this part was omitted from the problem.

At the present time, the national and international olympiads attract a very highly sophisticated type of candidate. So many problem sets and books are published annually that it is impossible to warm over some old chestnuts and expect the students not to see through the disguise. Those setting problems for olympiads have a tough time making up good ones. I hope this paper will indicate to prospective problem composers one of the ways in which this can be done.

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Almost Pythagorean Triples

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Recently I was squaring and adding terms of the sequence $5, 10, 15, 20, 25, 30, \ldots$. I noted that $10^2 + 15^2 = 18^2 + 1$, $20^2 + 25^2 = 32^2 + 1$, $25^2 + 35^2 = 43^2 + 1$. I kept getting a square plus one. This suggested solving the Diophantine equation:

$$x^2 + y^2 = z^2 + 1. \tag{1}$$

Let's call a solution of (1) in integers an **Almost Pythagorean Triple (APT)**. In analytic geometry (1) represents a hyperboloid of revolution of one sheet, a doubly ruled surface. The tangent plane at a point cuts this surface in two straight lines called rulings. There are two sets of rulings, and each line of one set meets no other line of that set, but meets every line of the other set except one line, to which it is parallel.

The transformation which sends the point (x, y, z) into $(-x, -y, -z)$ sends each ruling into a parallel ruling of the other set. We shall show that the integral solutions of (1) lie on such rulings, and that we can use this fact to find all APT's. With a table of squares you can make a list of APT's. The first six are: (5,5,7), (4,7,8), (8,9,12), (7,11,13), (11,13,17), (10,15,18). You may notice that integers with last digit 2,3,7,8 repeatedly appear as the z-coordinate of an APT. Those ending in 2 or 7 form an arithmetic progression, as do the corresponding x and y coordinates. Likewise with those ending in 3 or 8. These arithmetic progressions give us the formulas:

$$x = 3t + 2, \qquad y = 4t + 1, \qquad z = 5t + 2 \tag{2}$$

$$x = 3t + 1, \qquad y = 4t + 3, \qquad z = 5t + 3. \tag{3}$$

We see that (1) is satisfied when we substitute these formulas in it. Integral values of t in (2) and (3) give us infinitely many APT's. Formulas (2) and (3) are sets of parametric equations of two parallel lines with direction numbers $3, 4, 5$ (a primitive Pythagorean triple), lying in the surface (1).

Generalizing this, we show that if a, b, c is any primitive Pythagorean triple, that is, $\mathrm{GCD}(a, b, c) = 1$, then there are a pair of parallel rulings on (1) with direction numbers a, b, c, and that all APT's lie on such rulings.

First we find the direction numbers of the two rulings through any APT (p, q, r). If the parametric equations of a ruling through (p, q, r) are $x = At + p$, $y = Bt + q$, $z = Ct + r$, then $(At + p)^2 + (Bt + q)^2 = (Ct + r)^2 + 1$ for all t implies that:

$$A^2 + B^2 = C^2 \tag{4}$$

$$Ap + Bq = Cr. \tag{5}$$

Solving (4) and (5) for A, B, C in terms of p, q, r, we find the direction numbers A, B, C and A', B', C' of the two rulings.

$$\begin{aligned} A &= pr + q & A' &= pr - q \\ B &= qr - p & B' &= qr + p \\ C &= r^2 + 1 & C' &= r^2 + 1. \end{aligned} \tag{6}$$

These direction numbers are not necessarily primitive Pythagorean triples. We can get primitive triples from the formulas in (6) by dividing by the GCD. It is known that we can get all primitive Pythagorean triples from the formulas: $a = 2uv$, $b = u^2 - v^2$, $c = u^2 + v^2$, where u and v are relatively prime, and one is odd and the other even.

Given a primitive Pythagorean triple a, b, c we show how to write the equations of the two rulings on (1) with direction numbers a, b, c as follows:

$$\begin{array}{lll} x = at + p, & y = bt + q & z = ct + r \\ x = at + p' & y = bt + q' & z = ct + r', \end{array} \tag{7}$$

where $p + p' = a$, $q + q' = b$, $r + r' = c$. One can check that if one set of formulas in (7) satisfies (1), so does the other. Since from (6) c divides $r^2 + 1$, we have $r^2 \equiv -1 \pmod c$. This congruence will have two solutions r and r' with $r + r' = c$, since each prime factor of c is of the form $4n + 1$. See [3]. With one of these values of r we get from (5)

$$ap + bq = cr. \tag{8}$$

Solving this linear Diophantine equation for p and q, and finding p' and q' from $p + p' = a$, $q + q' = b$, we may write the equations (7) of the two rulings. Since every APT lies on such a ruling we can find them all. One can take the same approach to solve in integers the more general equation:

$$x^2 + y^2 = z^2 + s^2, \tag{9}$$

where s is a fixed integer.

Some APT's are isosceles, like (5,5,7). We can find all isosceles APT's by solving the Pell equation:

$$x^2 - 2y^2 = -1. \tag{10}$$

See [3]. Solutions of (10) are given by the even numbered convergents of the continued fraction for $\sqrt{2}$. The first three isosceles APT's are: (5,5,7), (29,29,41), (169,169,239). The odd numbered convergents give solutions of the other Pell equation:

$$x^2 - 2y^2 = 1. \tag{11}$$

These will give isosceles **Nearly Pythagorean Triples (NPT)**, that is, solutions of the Diophantine equation:

$$x^2 + y^2 = z^2 - 1. \tag{12}$$

Some examples of NPT's are: (2,2,3), (4,8,9), (12,12,17), (8,32,33), (10,50,51), (22,46,51), (34,38,51). Geometrically, equation (12) represents a hyperboloid of two sheets, which has no real rulings. The method used to find APT's does not work for NPT's. However, each APT determines an NPT, and conversely.

THEOREM. *If (p, q, r) is an APT, then $(2pr, 2qr, 2r^2 + 1)$ is an NPT, and if (p, q, r) is an NPT, then $(2p^2 + 1, 2pq, 2pr)$ is an APT.*

This is easily checked by substituting in equations (1) and (12). Applying this to the formulas for APT's given by equations (2) and (3) gives us the formulas for NPT's:

$$x = 30t^2 + 32t + 8, \quad y = 40t^2 + 26t + 4, \quad z = 50t^2 + 40t + 9 \tag{13}$$

$$x = 30t^2 + 28t + 6, \quad y = 40t^2 + 54t + 18, \quad z = 50t^2 + 60t + 19. \tag{14}$$

These formulas give us infinitely many NPT's. Similarly, the equations for any pair of parallel rulings on (1) give rise to formulas for NPT's. But not every NPT can be found in this way.

Equation (12) is the special case where $w = 1$ of the equation:

$$x^2 + y^2 + w^2 = z^2. \tag{15}$$

A complete solution of (15) was found by Catalan in 1885. See [1]. This solution is: $x = 2(pr + qs)$, $y = 2(qr - ps)$, $w = p^2 + q^2 - r^2 - s^2$, $z = p^2 + q^2 + r^2 + s^2$. If $w = 1$, then setting $r^2 + s^2 = n$, we get $p^2 + q^2 = n + 1$. In this case, a solution of (12), that is, an NPT, is given by $x = 2(pr + qs)$, $y = 2(qr - ps)$, $z = 2n + 1$. We can find an NPT whenever we have two consecutive integers n and $n + 1$, each of which is a square or the sum of two squares. A complete solution of (12) in rational numbers rather than integers is given by $(x/w, y/w, z/w)$, where x, y, z, w are given by Catalan's formulas.

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Simultaneous Triangle Inequalities

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It is a well-known result that the necessary and sufficient conditions that three positive numbers be the lengths of the sides of some triangle are that

$$b+c>a, \qquad c+a>b, \quad a+b>c. \tag{1}$$

Clearly it then follows that

$$(b+c-a)(c+a-b)(a+b-c)>0. \tag{2}$$

Also, it is easy to see that for $a, b, c>0$, (2) $\Rightarrow$ (1). For at most one of the three factors in (2) can be $\leqslant 0$ and this would violate (2). The latter inequality is also equivalent to

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c)>0$$

or, by multiplying out, to

$$2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)>0. \tag{3}$$

By Heron's formula for the area F of a triangle [**1**], (3) is given more compactly as

$$16F^2>0. \tag{3'}$$

As an extension of the above results, one can ask does there exist a polynomial inequality in the n positive numbers $a_1, a_2, \ldots, a_n$ which implies that any three of the numbers are lengths of sides of a triangle. Offhand one would expect that such a polynomial inequality exists and also that its degree is at least of order kn. Surprisingly, there is such a polynomial of degree 4 for all $n>3$. Formally, our result is as follows:

If $a_1, a_2, \ldots, a_n>0$ for $n \geqslant 3$ and

$$\left(a_1^2+a_2^2+\cdots+a_n^2\right)^2>(n-1)\left(a_1^4+a_2^4+\cdots+a_n^4\right), \tag{4}$$

then a_i, a_j, a_k, for all $i \neq j \neq k$, are lengths of sides of a triangle.

Our proof is by induction. First we show that (4) implies that

$$\left(a_2^2+a_3^2+\cdots+a_n^2\right)^2>(n-2)\left(a_2^4+a_3^4+\cdots+a_n^4\right), \tag{5}$$

where the left out term, a_1, is arbitrary. After some elementary algebra involved in completing a square, (4) can be shown to be equivalent to

$$0>\left\{a_1^2-S_2/(n-2)\right\}^2-\left\{S_2^2-(n-2)S_4\right\}\{n-1\}/\{n-2\}^2,$$

where $S_m=a_2^m+a_3^m+\cdots+a_n^m$. Consequently, (4) $\Rightarrow$ (5). Then by induction, rational numbers rather than integers is given by $(x/w, y/w, z/w)$, where x, y, z, w are given by Catalan's formulas.

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where $S_m=a_2^m+a_3^m+\cdots+a_n^m$. Consequently, (4) $\Rightarrow$ (5). Then by induction,

$$\left(a_i^2+a_j^2+a_k^2\right)^2>2\left(a_i^4+a_j^4+a_k^4\right),$$

which corresponds to (3).

Inequality (4) arose in my generalization of problem 1087, *Crux Mathematicorum*, 11(1985) 289, i.e., there exists a regular $(n-1)$-dimensional simplex $A_1A_2\cdots A_n$ of edge length a and a point P in its space such that $PA_i=a_i$, $i=1,2,\ldots,n$, if and only if inequality (4) holds.

Note that by applying Ptolemy's inequality in 3-space to the tetrahedron $P\text{-}A_iA_jA_k$, it follows that a_i, a_j, a_k are lengths of sides of a triangle.

It is to be noted that whereas (3) is a necessary and sufficient condition on three positive numbers to be the lengths of sides of a triangle, (4) is only a sufficient condition that any three of n positive numbers are lengths of sides of a triangle. For example, consider the four numbers 5, 5, 5, and 9. Any three of them are lengths of sides of a triangle, but (4) for $n=4$ is not satisfied. As an open problem, find a polynomial inequality which is both a necessary and sufficient condition for the latter property. Finally, as another open problem, find a polynomial inequality on n positive numbers $a_1, a_2, \ldots, a_n$ such that any r of them, with $n>r>3$, are lengths of sides of an r-gon.

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On the Sum of Consecutive *K*th Powers

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In the early 18th century, James Bernoulli discovered an elegant formula for the sum of consecutive Kth powers:

$$1^K+2^K+\cdots+(n-1)^K=\sum_{i=0}^{K}\binom{K}{i}B_i\frac{n^{K+1-i}}{K+1-i}\qquad(K=1,2,\ldots).\tag{1}$$

Here, $B_0, B_1, B_2, \ldots$ are the so-called *Bernoulli numbers* which arise as coefficients in the power series expansion of $x/(e^x-1)$:

$$\frac{x}{e^x-1}=\sum_{K=0}^{\infty}\frac{B_K}{K!}x^K\quad(\text{valid for }|x|<2\pi).$$

The first few values of B_K are

$$B_0=1,\qquad B_1=-\frac{1}{2},\qquad B_2=\frac{1}{6},$$

$$B_3=0,\qquad B_4=-\frac{1}{30},\qquad B_5=0.$$

It is well known that if K is odd and greater than 1, then $B_K=0$, while if K is even and greater

$$\left(a_i^2+a_j^2+a_k^2\right)^2>2\left(a_i^4+a_j^4+a_k^4\right),$$

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Note that by applying Ptolemy's inequality in 3-space to the tetrahedron $P\text{-}A_iA_jA_k$, it follows that a_i, a_j, a_k are lengths of sides of a triangle.

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$$B_3=0,\qquad B_4=-\frac{1}{30},\qquad B_5=0.$$

It is well known that if K is odd and greater than 1, then $B_K=0$, while if K is even and greater than 0, then the B_K alternate in sign (see, for example, [**1**]). For $K=1,2,3$ we obtain the familiar identities

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Bernoulli never offered a proof of (1). In his posthumously printed work *Ars Conjectandi* (1713), he simply wrote down the formula for the first ten values of K and then claimed that the general pattern was apparent!

The classical proof of Bernoulli's identity is usually based either on the "Euler-Maclaurin Summation Formula" [**2**] or on the "Bernoulli polynomials" [**1**]. In this note we offer a far more direct and completely elementary proof.

We begin with the geometric sum

$$1+r+r^2+\cdots+r^{n-1}=\frac{r^n-1}{r-1}$$

in which we take $r=e^x$; the result is

$$1+e^x+e^{2x}+\cdots+e^{(n-1)x}=\frac{e^{nx}-1}{x}\cdot\frac{x}{e^x-1}.$$

The trick is to expand each side as a power series in x and then equate coefficients of x^K. The left side is easy; using the Maclaurin series for each exponential and then adding the results, we obtain

$$n+\sum_{K=1}^{\infty}\frac{1^K+2^K+\cdots+(n-1)^K}{K!}x^K. \tag{2}$$

The right side, on the other hand, is just

$$\left(\sum_{K=0}^{\infty}\frac{n^{K+1}}{(K+1)!}x^K\right)\left(\sum_{K=0}^{\infty}\frac{B_K}{K!}x^K\right). \tag{3}$$

Multiplying together the two series in (3) and then equating coefficients in (2), we find that $B_0=1$ and that for $K\geqslant 1$

$$\frac{1^K+2^K+\cdots+(n-1)^K}{K!}=\sum_{i=0}^{K}\frac{n^{K+1-i}}{(K+1-i)!}\frac{B_i}{i!}.$$

The proof is over—simply multiply both sides by $K!$.

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The "Ladder Problem"

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A good mathematical problem is like a good song. It keeps coming back time and time again. And just as a song, when it comes back periodically, is still the same song—just sung by someone new in a different style—so a problem often reappears to be proposed in a different framework and/or solved in a different way.

Such is the case of the problem I have come to call, affectionately, the "ladder problem." This problem was originally posed to me by an engineer who said it was being "kicked around" in various engineering circles. I don't know its original source; it may be lost in antiquity.

Before I pose the problem, I would like to ask, "What *is* a 'good' problem?" Of course, that is open to numerous opinions, but surely it would meet several of the criteria listed below and, perhaps, these criteria could be used in evaluating candidates.

1. It can be described quite simply and the crux of the problem can be quickly grasped by one with even minimal mathematical background.
2. It can be nicely cast in a physical, tangible framework.
3. It immediately intrigues and challenges people to whom it is posed.
4. It can be solved in a creative, refreshing and elegant manner, as well as in, perhaps, a more straightforward, but uninteresting way.
5. It can be solved in a variety of ways.
6. It can be solved without using "higher" mathematics.
7. It quickly suggests a variation or generalization which results in an even more challenging problem.

The "ladder problem," I believe, excels in all these categories. Here is the problem as it was posed to me:

> A 16-ft ladder is leaning against a wall in such a manner (FIGURE 1) that one point of the ladder is just touching a box or other obstruction which has a 4-ft by 4-ft cross section and is pushed against the wall. How much of the ladder is between the wall and the point of contact and how much is between the point of contact and the floor?

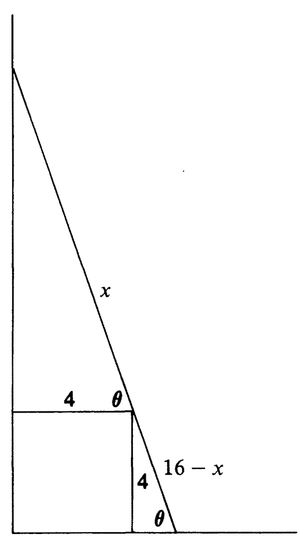


FIGURE 1

"Oh, this looks easy," I thought, as did many others with whom I have shared the problem over the years, and I plunged right in. It didn't take me long to generate a 4th-degree equation—something nearly all would-be solvers quickly do—for by the Pythagorean theorem and similar triangles one can quickly arrive at the proportion

$$4/(16-x)=\sqrt{x^2-16}/x$$

which, after squaring both sides, clearing of fractions and simplifying leads to the equation

$$x^4-32x^3+224x^2+512x-4096=0.$$

But I knew that a general 4th-degree equation could only be solved by cumbersome formulas or approximated by some approximation technique. Neither of these appealed to me and my intuition kept insisting there was another way. However, several hours later I laid down my pencil, now several inches shorter, and gave up—at least temporarily.

Soon I, like others after me, took it up again. And again. Eventually (my pride won't allow me to reveal how eventually), I solved it by some tricky trigonometric maneuvers. I raced to show my solution to my colleagues who by now were also "hooked" on the problem. Here is my solution:

(1) From the similar triangles one can quickly obtain the equations

$$\cos\theta=4/x$$
$$\sin\theta=4/(16-x).$$

(2) After solving each of these for x, equating, and a little manipulation, we get the equation

$$\sin\theta+\cos\theta=4\sin\theta\cos\theta.$$

(3) Now square both sides and apply the identities $\sin^2\theta+\cos^2\theta=1$ and $\sin 2\theta=2\sin\theta\cos\theta$. Presto! An equation which is quadratic in $\sin 2\theta$ appears,

$$4\sin^2 2\theta-\sin 2\theta-1=0.$$

(4) Applying the quadratic formula we find that

$$\sin 2\theta=(1+\sqrt{17})/8 \quad \text{(the other root may be discarded)}.$$

Since $0° < \theta < 90°$ or $0° < 2\theta < 180°$, there are two possible solutions to the above equation. They are

$$\theta_1=\sin^{-1}((1+\sqrt{17})/8)/2 \doteq 19.91°$$

and

$$\theta_2=(180-\theta_1)/2 \doteq 70.09°.$$

Actually these represent the "same" solution reflecting the fact that the ladder can be positioned in two ways (just switch the similar triangles). These lead to x values of

$$x=4/\cos\theta_1 \doteq 4.25$$

and

$$x=4/\cos\theta_2 \doteq 11.75.$$

and this shows how the ladder is divided.

My colleagues were sufficiently impressed, and I laid the problem to rest although each year I faithfully resurrected the problem and posed it to my trigonometry class after we had discussed trigonometric identities and equations. I would demonstrate the problem by placing a rectangular wastebasket or a stack of books against a wall and then show them the two ways a meter stick could be positioned so that it touched the floor, the wall, and the wastebasket. For years no one was able to solve it, although several students came up with very accurate approximations. I was disappointed and yet, secretly pleased, for it meant the problem still "belonged" to me. But finally, a student did solve it and to my amazement, without the use of trigonometry.

I swallowed my pride, congratulated him, and settled back to celebrate in the simple beauty of his solution:

(1) He started with the similar triangle proportion mentioned earlier,

$$4/(16-x)=\sqrt{x^2-16}\,/x.$$

(2) Squaring both sides and then making the clever and insightful substitution, $x=8-y$, he transformed a general 4th-degree equation into a special one.

$$16/(8+y)^2=\left((8-y)^2-16\right)/(8-y)^2.$$

(3) Expanding, clearing of fractions and simplifying leads to the equation

$$y^4-160y^2+2048=0,$$

which is quadratic in y^2.

(4) Applying the quadratic equation *twice* produces

$$y=\pm 4\sqrt{5\pm\sqrt{17}}$$

or

$$x=8\pm 4\sqrt{5\pm\sqrt{17}}\ .$$

(5) A little investigation easily shows that the two relevant solutions are

$$8+4\sqrt{5-\sqrt{17}}\doteq 11.75$$

and

$$8-4\sqrt{5-\sqrt{17}}\doteq 4.25.$$

Never one to be outdone, I continued to look for a "better" solution. I have never found one, but very recently I did discover *another* one. I started by realizing that this problem, stripped of a physical setting could be posed as follows:

> Find the intercepts of a line which passes through (4,4) if the distance between the intercepts is 16. (FIGURE 2)

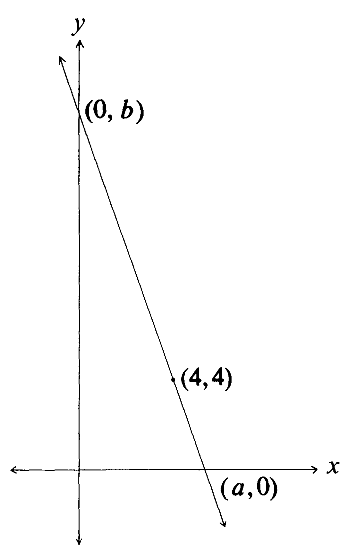


FIGURE 2

Using the same idea of similar triangle proportions one can easily derive the equation $4a+4b=ab$. This equation and the equation $a^2+b^2=16$ form a system which is solvable, although the solution is not pleasant and I won't trouble you with it.

However, this way of looking at the problem did lead me to a realization which is rather fascinating. The same problem can be even more simply described in the following way:

Find the sides of a rectangle if the area is twice the perimeter and the diagonal is 16. (FIGURE 3).

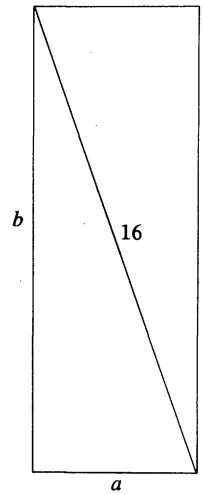


FIGURE 3

This follows immediately from the two equations above if one makes the "ladder" the diagonal of a rectangle with sides of a and b.

As a final note, the alert reader may notice that the solutions presented are possible *only* because of a special characteristic of the obstruction—it has a square cross section. Can anyone solve the problem if the obstruction has, perhaps, a 3-ft by 5-ft cross section? I doubt it—but I've said that before.

Let us remember humility else we be immortalized:

"Young Arthur [Conan Doyle] went to Stonyhurst College in Lancashire... If the name Holmes came from Oliver Wendell, the Sherlock came from a Patrick Sherlock, who was the dullest of all Arthur's contemporaries at Stonyhurst. There were two Moriartys—John Francis and Michael—both of whom won the Stonyhurst prize for mathematics."

Burgess, A., *The Sainted Sleuth*, *Still on the Case*, The New York Times Book Review, January 4, 1987, p. 1.

PROBLEMS

Loren C. Larson, Editor
Bruce Hanson, Associate Editor
St. Olaf College

Proposals

To be considered for publication, solutions should be received by April 1, 1988

1272. *Proposed by Jerrold W. Grossman, Oakland University, Michigan, and Allen J. Schwenk, Western Michigan University.*

For n a positive integer, let $f(n)$ be the smallest prime number p such that $n+p$ is also prime, if such a p exists, and 0 otherwise. Show that f is unbounded.

1273. *Proposed by Benjamin G. Klein, Davidson College, North Carolina, and John Layman, Virginia Polytechnic Institute and State University.*

Let q be a positive integer and let p be a prime. Let $f_0 = f_1 = f_2 = p$ and define f_n for $n > 2$ by

$$f_{n+1} = (f_n f_{n-1} + q)/f_{n-2}.$$

a. Prove that if p is odd, then f_n is an integer for all n if and only if p^2 divides q.
b. Prove that if $p = 2$, then f_n is an integer for all n if and only if p divides q.

1274. *Proposed by S. Kung, Jacksonville University, Florida.*

Let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). If $f(a) = f(b) = 0$ and $f(x) > 0$ for $x \in (a, b)$, show that for any positive real number r there is a number $c \in (a, b)$ such that $rf'(c) + f(c) = 0$.

1275. *Proposed by Fouad Nakhli, American University of Beirut.*

Let c be a positive real number, let $a_1 = c$ and for $n \geqslant 1$, $a_{n+1} = c^{1/a_n}$.
a. Show that the sequence (a_n) converges if $0 < c \leqslant 1$.
b. Show that the series $\sum_{n=1}^{\infty} a_n$ converges if $0 < c < (1/e)^{1/e}$.

ASSISTANT EDITORS: Clifton Corzatt and Theodore Vessey, *St. Olaf College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren C. Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

1276. *Proposed by Lawrence J. Wallen, University of Hawaii.*

Let A be the $(n+1)\times(n+1)$ symmetric matrix whose (i,j)th entry is $\binom{i+j}{i}$, $i,j=0,1,2,\dots,n$. Let the characteristic polynomial $p(\lambda)$ of A be

$$a_0\lambda^{n+1}+a_1\lambda^n+a_2\lambda^{n-1}+\cdots+a_{n+1}\qquad(a_0=1).$$

Show that

$$a_j=(-1)^{n+1}a_{n+1-j},\qquad j=0,1,\dots,n+1.$$

Quickies

Q724. *Proposed by David Singmaster and Heather Vaughan, Polytechnic of the South Bank, London, England.*

Describe how to find all positive integers x, y, z such that $x^2+y^2=z^2\pm 1$.

Q725. *Proposed by S. Kung, Jacksonville University, Florida.*

Show that

$$(\sin x)\,y<\sin xy,\qquad 0<x<\pi\qquad 0<y<1.$$

Solutions

A Combinatorial Partition **October 1986**

1247. *Proposed by Constantin Gonciulea, Traian College, Drobeta Turnu-Severin, Romania.*

Let a and b be positive integers. Suppose that A and B are finite and disjoint sets of integers such that if $i\in A\cup B$ then $i+a\in A$ or $i-b\in B$. Show that $a|A|=b|B|$. (If X is a finite set, $|X|$ denotes the number of elements in X.)

I. *Solution by the Mathematical Problem Solving Group, Shippensburg University, Pennsylvania.*

Define a map $f\colon A\cup B\to A\cup B$ by

$$f(i)=\begin{cases} i+a & \text{if } a+i\in A\\ i-b & \text{otherwise.}\end{cases}$$

Using the fact that $A\cap B=\varnothing$, one can easily show that f is one-to-one. Since $A\cup B$ is finite, it follows that f is a permutation of $A\cup B$.

Write f as a permutation of k disjoint cycles, and let $m_1, m_2,\dots,m_k$ be representatives of the orbits of each of the k cycles of f.

Observe that $f^{|\mathrm{orb}(m_i)|}$ is the identity map on $\mathrm{orb}(m_i)$, the orbit of m_i, and therefore

$$m_i+a|A\cap\mathrm{orb}(m_i)|-b|B\cap\mathrm{orb}(m_i)|=m_i,$$

or equivalently,

$$a|A\cap\mathrm{orb}(m_i)|=b|B\cap\mathrm{orb}(m_i)|.$$

It follows that

$$a|A|=a\sum_{i=1}^{k}|A\cap\mathrm{orb}(m_i)|=b\sum_{i=1}^{k}|B\cap\mathrm{orb}(m_i)|=b|B|.$$

II. *Solution by Bjorn Poonen, student, Harvard College.*

Let $S_A = \{n - a : n \in A\}$, $S_B = \{n + b : n \in B\}$, and $S = S_A \cup S_B$. Then $A \cup B \subseteq S$. But $|S| \leqslant |A| + |B|$, so $S = A \cup B$, and S_A and S_B must be disjoint. The sum of the elements in $S_A \cup S_B$ is $-a|A| + b|B|$ more than the sum of the elements in $A \cup B$, so $a|A| = b|B|$.

Also solved by Peter Andrews and Edward T. H. Wang (Canada), S. F. Barger, Duane M. Broline, Robert Doucette (student), Roger B. Eggleton (Australia), Griff Elder, Antonio Jose Duran Guardeño (Spain), Ram Prakash Gupta (Virgin Islands), Thomas Hofmeister (West Germany), M. Riazi-Kermani, Matt Richey, J. M. Stark, William P. Wardlaw, Eric Wepsic (student), Western Maryland College Problems Group, K. D. Wallace and R. G. Powers, and the proposer.

A Curious Property of 1/7 **October 1986**

1248. *Proposed by Edward Kitchen, Santa Monica, California.*

From $1/7 = .1428571\ldots$, we get the points (1,4), (4,2), (2,8), (8,5), (5,7), and (7,1). Show that they lie on an ellipse.

I. *Solution by John C. Nichols, Thiel College, Pennsylvania.*

Substitute the coordinates of the six points into the equation of a general conic, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, and solve for the coefficients. This yields

$$19x^2 + 36xy + 41y^2 - 333x - 531y + 1638 = 0.$$

This is an ellipse, since the discriminant, $B^2 - 4AC$, is negative.

II. *Solution by G. J. Griffith, University of Saskatchewan, Saskatoon, Canada.*

Let j_i be the ith digit in the decimal expansion of 1/7 and let $P_i \equiv (j_i, j_{i+1}, 1)$, (homogeneous coordinates).

For $1 \leqslant i \leqslant 6$, let ℓ_i be the line $P_i P_{i+1}$.

It is easy to check that lines ℓ_i and ℓ_{i+3}, $i = 1,2,3$, intersect on the (ideal) line $z = 0$.

Since the opposite sides of the hexagon $\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6$ intersect in three collinear points, the hexagon may be inscribed in a conic (the converse of Pascal's Theorem).

Also solved by Charles Ashbacher, Anders Bager (Denmark), Seung Jin Bang (Korea), J. C. Binz (Switzerland), J. E. Breneman, Duane Broline, Hüseyin Demir (Turkey), Thomas P. Dence, Ragnar Dybvik (Norway), David Earnshaw (Canada), Milton P. Eisner, Irwin K. Feinstein, Alex G. Ferrer (Mexico), Nick Franceschine, Michael Froning, Herta T. Freitag, Ramesh Garimella and Russell Euler, G. J. Griffith (Canada, second solution), Cornelius Groenewoud, J. T. Groenman (Netherlands), Ram Prakash Gupta (Virgin Islands), Paul M. Harms, P. L. Hon (Hong Kong), John P. Hoyt, Geoffrey A. Kandall, Vaclav Konecny, Gene Arnold and Philip Stich, L. Kuipers (Switzerland), Kee-wai Lau (Hong Kong), Marvin Marcus and Claire Pesce, Roger B. Nelsen, Bjorn Poonen (student), M. Riazi-Kermani, Rodney R. Saunders, Harvey Schmidt, Jr., The Shippensburg University Problem Solving Group, Donald P. Skow, Stephanie Sloyan, Robert S. Stacy (West Germany), J. M. Stark, David Stone, Arthur Sparks, Walter Lynch and Ginger Bolton, R. W. W. Taylor, W. R. Utz, Michael Vowe (Switzerland), William P. Wardlaw, Harry Weingarten, Staffan Wrigge (Sweden), and the proposer.

Several readers noted that the six points determined by $1/13 = .0769230\ldots$ lie on a hyperbola. Froning noted that the other 6-digit repetend of the 13ths ($2/13 = .1538461\ldots$) produces another hyperbola. Another ellipse is produced by $1/39 = .0256410\ldots$, but no conic is generated by $5/39 = .1282051\ldots$, nor by $7/39 = .1794871\ldots$. The Shippensburg University Problem Solving Group investigated all period six reciprocals, and found that the reciprocals of 13 and 77 lie on hyperbolas, reciprocals of 39, 63, 91, 143, 273, 429, 693, and 819 lie on ellipses, and reciprocals of 21, 117, 189, 231, 259, 297, 351, 407, 481, and 777 do not lie on a conic. All other period six reciprocals are on conics because they have fewer than enough points to determine a conic uniquely.

1249. *Proposed by Victor Hernandez, Universidad Autónoma de Madrid, Spain.*

a. Find all continuous functions f that satisfy:

$$f(f(x))=f(x), \quad \text{for all real } x. \tag{1}$$

b. Find all differentiable functions that satisfy (1).

Solution by John O. Kiltinen, Northern Michigan University.

(a) We prove that a continuous function f from $\mathbb{R}$ to $\mathbb{R}$ satisfies condition (1) if and only if it satisfies the two conditions:

(i) When restricted to its range $f(\mathbb{R})\equiv I$, f coincides with the identity function on I, and

(ii) The range I is a closed interval (possibly unbounded).

Clearly, (i) is a transparent restatement of (1), and thus (1) and (i) are equivalent without regard to the continuity of f. Thus, to prove the claim, it only remains to show that (ii) follows from (1) if f is continuous.

From well-known facts of topology, since $\mathbb{R}$ is connected, so is $I=f(\mathbb{R})$, and thus, I must be an interval. (Those not familiar with topology can use the Intermediate Value Theorem to show that I is an interval.) To complete the proof, we need to show that if I has an endpoint a, then $a\in I$. Suppose that a is a left endpoint of I. If $I=[a,a]$, then clearly $a\in I$; otherwise I has positive length and contains elements to the right of a. By the continuity of f, $f(a)=\lim_{x\to a^+}f(x)$, so $f(a)=f(\lim_{x\to a^+}x)=\lim_{x\to a^+}f(x)=\lim_{x\to a^+}x=a$. The second equality in this string is justified by the continuity of f and the third by the fact that elements immediately to the right of a are in I, and $f(x)=x$ if $x\in I$. Thus, $f(a)=a$, so $a\in I$. Similarly, a right endpoint of I is in I.

(b) A differentiable function f satisfies (1) if and only if f is the identity function or a constant function.

To see this, we will show that under the given hypotheses, I must either be all of $\mathbb{R}$ or a single point. These restrictions, together with (i), force f to be either the identity function or a constant function.

Suppose that I has a left endpoint a and that I has nonzero length. Then for x near a and to the right, $f(x)=x$, so

$$\lim_{x\to a^+}\frac{f(x)-a}{x-a}=1.$$

However, for x to the left of a, $x-a<0$ but $f(x)\geqslant a$, so $f(x)-a\geqslant 0$. Thus

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$$

is at most zero if it exists, and so the left and right derivatives of f at a cannot be equal. Thus, I cannot have a left endpoint unless I is a single point, and the same can be shown for right endpoints.

Also solved by Irl C. Bivens and Benjamin G. Klein, Duane Broline, Douglas Cashing and Charles Diminnie, Julio Castiñeira (Spain), Chico Problem Group, D. K. Cohoon, Michael W. Ecker, W. O. Egerland and C. E. Hansen, Griff Elder, Ramesh Garimella and Russell Euler, Michael B. Gregory, Antonio Jose Duran Guardeño (Spain), H. Peter Gumm (West Germany), Ram Prakash Gupta and Amir A. Maleki (Virgin Islands), G. A. Heuer, Thomas Hafmeister (West Germany), Humboldt State University '86 Proof Group, M. Nijmeijer (The Netherlands) and M. Staring, Gene M. Ortner, Bjorn Poonen (student), M. Riazi-Kermani, Harry Sedinger, Shippensburg University Mathematical Problem Solving Group, Dennis Spellman, David C. Vella, William P. Wardlaw, Gregory P. Wene, Eric Wepsic (student),. Western Maryland College Problems Group, Westmont College Problem Solving Group, and the proposer. There was one incomplete solution.

G. M. Ewing and W. R. Utz have shown that the continuous solutions of (1) of this problem are the same as the continuous solutions of $f^{(n)}=f$ for all even $n\geqslant 2$. (See their paper, "Continuous solutions of the functional equation $f^n(x)=f(x)$", *Canadian Journal of Mathematics*, (5) 1953, pp. 101–103.)

The Pellian Sequence October 1986

1250. *Proposed by J. Metzger, University of North Dakota, and S. Kaler, Honeywell Corporation.*

The Pellian sequence $(x_n)_{n=1}^{\infty}$ is defined as follows: x_n is the smallest positive integer x for which nx^2+1 is a perfect square; if no such x exists, x_n is defined to be 0. The sequence begins $0,2,1,0,4,2,3\dots$.

a. Show that every positive integer occurs infinitely often in the Pellian sequence.

b. Determine all occurrences of p^k, p a prime, $k>0$, in the Pellian sequence.

Solution by M. Riazi-Kermani, Fort Hays State University, Hays, Kansas.

(a) Let i be a positive integer. We need to show that i occurs infinitely often in (x_n).

If $i=1$, let $n=m^2-1$, $m=2,3,4,\dots$. Then $1+n=m^2$, so $x_n=1$.

If $i>1$, let $n=m(mi^2-2)$, $m=1,2,3,\dots$. Then $1+ni^2=(mi-1)^2$, and if $0<x<i$, $(mix-1)^2<1+nx^2<(mix)^2$. Thus, $1+nx^2$ is not a perfect square, so $x_n=i$.

(b) First consider the case of an odd prime p. We show that $x_n=p^k$, $k>0$, if and only if $n=m(mp^{2k}\pm2)$, $m=1,2,3,\dots$.

For $n=m(mp^{2k}\pm2)$, $1+np^{2k}=(mp^{2k}\pm1)^2$. Furthermore, if $0<x<p^k$,

$$\left(mp^kx\right)^2<1+nx^2<\left(mp^kx+1\right)^2 \quad \text{for } n=m\left(mp^{2k}+2\right),$$

and

$$\left(mp^kx-1\right)^2<1+nx^2<\left(mp^kx\right)^2 \quad \text{for } n=m\left(mp^{2k}-2\right).$$

These inequalities show that $1+nx^2$ is not a perfect square, so therefore, $x_n=p^k$.

On the other hand, suppose that $x_n=p^k$, $k>0$. Then $1+np^{2k}=t^2$ for some positive integer t. Therefore p^{2k} divides t^2-1. Since $t^2-1=(t+1)(t-1)$ and p can divide only one of $t+1, t-1$, it must be the case that p^{2k} either divides $t+1$ or $t-1$. Thus, t has the form $mp^{2k}\pm1$ for some positive integer m, and it follows that $n=m(mp^{2k}\pm2)$.

Now consider the case $p=2$. We show that $x_n=2^k$, $k>0$, if and only if

$$n=\begin{cases} m\left(m2^{2k-2}+1\right) & m=1,2,3,\dots \\ m\left(m2^{2k-2}-1\right) & m=2,3,4,\dots. \end{cases}$$

If n has this form, then $1+n2^{2k}=(m2^{2k-1}\pm1)^2$. Also, for $0<x<2^k$,

$$\left(2^{k-1}mx\right)^2<1+nx^2<\left(2^{k-1}mx+1\right)^2 \quad \text{for } n=m\left(m2^{2k-2}+1\right),$$

and

$$\left(2^{k-1}mx-1\right)^2<1+nx^2<\left(2^{k-1}mx\right)^2 \quad \text{for } n=m\left(m2^{2k-2}-1\right).$$

It follows that $1+nx^2$ is not a perfect square, and $x_n=2^k$.

On the other hand, suppose that $x_n=2^k$. Then $1+n2^{2k}=s^2$ for some positive integer s. Thus, 2^{2k} divides s^2-1. Since $s^2-1=(s-1)(s+1)$, 2^{2k-1} divides either $s-1$ or $s+1$. Therefore, $s=m2^{2k-1}\pm1$ for some positive integer m. From this we find that $n=m(m2^{2k-2}\pm1)$. However, if $m=1$ then $1+n$ is a perfect square so that $x_n=1$. Thus we are left with the forms for n given above, and the solution is complete.

Also solved by J. C. Binz (Switzerland), Duane M. Broline, Ramesh Garimella and Russell Euler, Bjorn Poonen (student), Harvey Schmidt, Jr., and the proposers.

A Generalized Catalan Number October 1986

1251. *Proposed by Harry Ruderman, Lehman College, The Bronx, New York.*

Prove that

$$\frac{(3n)!}{n!(n+1)!(n+2)!}$$

is an integer for $n\geqslant3$.

Solution by Bjorn Poonen, student, Harvard College.

Let $[x]$ denote the greatest integer function.

Claim 1. $[3x]=[x]+[x+1/3]+[x+2/3]$ for all $x \geqslant 0$.

Proof. It is true for $x \in [0,1/3)$, and both sides of the equation are increased by 1 when x is increased by $1/3$.

Claim 2. For $p^m \geqslant 3$,

$$[3n/p^m] \geqslant [n/p^m]+[(n+1)/p^m]+[(n+2)/p^m].$$

Proof. This follows from substituting $x=n/p^m$ in Claim 1 and comparing terms.

The exponent of the prime p in the given expression is

$$\sum_{m=1}^{\infty}([3n/p^m]-[n/p^m]-[(n+1)/p^m]-[(n+2)/p^m]).$$

For $p \neq 2$, this is nonnegative by Claim 2.

Now we can pick k such that $n+2<2^k \leqslant 3n$, provided that $n \geqslant 3$. Then the exponent of 2 in the given expression is

$$\begin{aligned}\sum_{m=1}^{\infty}&([3n/2^m]-[n/2^m]-[(n+1)/2^m]-[(n+2)/2^m])\\ &\geqslant [3n/2]-[n/2]-[(n+1)/2]-[(n+2)/2]+[3n/2^k] \quad \text{(by Claim 2)}\\ &\geqslant f(n),\end{aligned}$$

where $f(n)=[3n/2]-[n/2]-[(n+1)/2]-[(n+2)/2]+1$. But $f(0)=f(1)=0$ and $f(n+2)=f(n)$, so $f(n)=0$ for all n. This completes the proof.

Also solved by Beno Arbel (Israel), Nicolas K. Artemiadis (Greece), David C. Bonner, Duane M. Broline, Thomas Hofmeister (West Germany), John O. Kiltinen, Kee-wai Lau (Hong Kong), Aron Pinker and Edieal J. Pinker (student), M. Riazi-Kermani, Shippensburg University Mathematical Problem Solving Group, J. M. Stark, Richard Vrem, Michael Vowe (Switzerland), and the proposer.

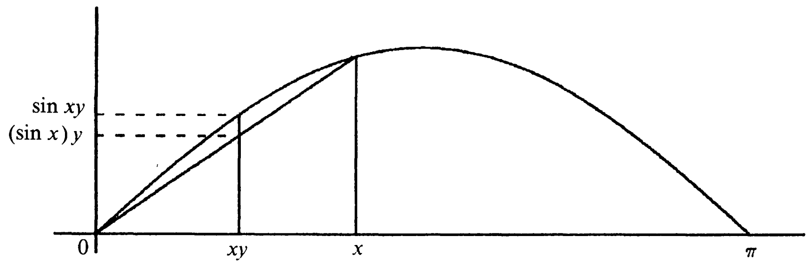
Mindful of the formula for the nth Catalan number, $(2n)!/n!(n+1)!$, one might expect the integer of our problem to count something, but no solutions hinted at this possibility. However, along these lines, John O. Kiltinen showed that $(4n)!/n!(n+1)!(n+2)!(n+3)!$ is an integer in all cases except when $n=2^k-1$, in which case it is a half-integer (i.e., its denominator is 2).

Answers

A724. Rewrite the equation as $x^2 \pm 1 = z^2 - y^2$. For any $x>0$, factor $x^2 \pm 1$ as ab, where $a \equiv b \pmod 2$ and $a>b$. Then y and z are given by $y=(a-b)/2$ and $z=(a+b)/2$.

(Note from the authors: This solution is based on factoring. Is there a simpler parametric solution? We have found only one paper on such triples—item D16-32 in Leveque's *Reviews in Number Theory*, and it involves generation by use of matrices.)

A725. *Proof without words.*



Similarly, if f is any concave function on $[0,a]$ such that $f(0)=0$, then $f(x)y<f(xy)$ for $0<x<a$, $0<y<1$.

REVIEWS

Paul J. Campbell, Editor
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Hewlett-Packard. HP-28C Scientific Professional Calculator, with *Getting Started Manual* (254 pp) and *Reference Manual* (406 pp). $175 wholesale. Technical descriptive articles in *Hewlett-Packard Journal* 38:8 (August 1987).

> This multifaceted calculator offers the usual features plus symbolic algebra, function plots (on a 4-line LCD display), an equation solver, numerical integration, and limited symbolic calculus. That's great! However, the symbolic integration is available only for polynomials, and Taylor series easily outrun the puny workspace memory; the muMath of 7 years previous had more capability. Users who find the HP-28C too confining—or perhaps too hard to learn to use from the 670 pp of manuals—may find themselves better served by a machine like the TRS-100. The latter, which is available in successor models now, offered great portability, a standard keyboard, and a versatility of software (word processing, muMath, communications, etc.). Yet many will find the HP-28C a handy calculating aid, and the technical articles on how the symbolic calculation and equation solving are done should be read by data structures and numerical analysis students respectively.

Maor, Eli, *To Infinity and Beyond: A Cultural History of the Infinite*, Birkhäuser, 1987; xvi + 275 pp, $49.50.

> Splendid and lavishly designed and illustrated book for the general reader about all aspects of infinity: mathematical, geometric, esthetic, cosmological. This book is destined to become a classic—may it appear in paperback some day soon, at your town's bookstore, and stay in print into infinity.

Rubik's Magic Puzzle: Link the Rings, Matchbox Toys, 1987, $8.99. Books for Aficionados: James G. Nourse, *Simple Solutions to Rubik's Magic*, Bantam, 1986, 64 pp, $2.95; Ashwin Belur and Blair Whitaker, *Rubik's Magic: The Solution*, Dell, 1986, 32 pp, $2.95; Christophe Bandelow, *Rubik's Magische Ringe*, Falken-Verlag, 1986, 96 pp, DM 6.80.

> Can lightening strike twice in the same place? Will this new puzzle catch on like Rubik's cube? Is it harder or easier? Was it really Rubik who invented it, or is it just being marketed under his name? Can mathematician Nourse sell 7 million copies of his booklet, as he did with the one on the cube? Sadly, by now you probably can't find this puzzle in your stores, much less the books. It is no less a classic for that, both in technical design (which Bandelow explains best) and in puzzlement.

Grinstein, Louise S., and Paul J. Campbell (eds.), *Women of Mathematics: A Biobibliographic Sourcebook*, Greenwood, 1987; 292 pp, $45.

> Let me begin by telling you upfront that I co-edited this book. If you are seeking an objective opinion on it, check another journal. I could have sent it to a colleague to

review (which would smack of cronyism) or had no review here (unthinkable for such a fine and important book—oh-oh, there goes any pretense of objectivity!). For each of 43 women of today and yesteryear, the book provides a biography, a non-technical account of the person's work, and bibliographies of works by and works about the individual. The editors specifically disclaim that these are the 43 best women mathematicians or some "representative" sample. They are, however, people with interesting lives, whom the authors and editors hope will inspire young women to pursue careers in mathematics, despite whatever obstacles are put in their paths.

Closs, Michael P. (ed.), *Native American Mathematics*, U Texas Pr, 1986; 431 pp, $35.

The ethics of reviewing one's own book are fairly simple (don't do it or else be blatant about it); more difficult is the matter of reviewing a book from which one's contribution was deleted. Mine had been accepted but (after several years passed) was deleted for space reasons. Had the book been typeset or printed on a laser printer instead of just being typed, the same page could have accommodated 60% more material. My article will appear elsewhere, and my own current experience at trying to prepare camera-ready copy for a book makes me very sympathetic to the editor. So much for the personally subjective part of this review; on to the merits of the book, which are considerable for those who are interested in the development of mathematics in culture. Twelve of the 13 essays are concerned with number systems, number words, and numerical representations used by native peoples of the American continents; the last essay considers Mesoamerican geometry in the context of building site-plans and archaeoastronomy. Like Claudia Zaslavsky's *Africa Counts* (1974), this book affords an opportunity and background for indigenous mathematics to become a standard part of study of other cultures.

Mac Lane, Saunders, *Mathematics: Form and Function*, Springer-Verlag, 1986; xi + 476 pp, $42.

Is their no refuge for a reviewer with ethical problems? Let honesty prevail again: here is a book about the nature of Mathematics (for so he capitalizes it) by my academic grandfather. One wishes for a general audience for this book, but the notational and conceptual demands that it makes suit it mainly for use in senior seminars for mathematics majors and foundations courses for preservice and inservice teachers, as he is able to survey and put into perspective so much of the mathematics that is segregated into different courses. Mac Lane turns from asking about mathematics "Is it true?" to asking if it is correct? responsive? illuminating? promising? relevant? Perhaps more surprisingly, he utterly rejects platonism: "[S]ave for mythology, all the variants of Platonism shatter on the actual practice of Mathematics . . . Mathematics aims to understand, to manipulate, to develop, and to apply those aspects of the universe which are formal. This view . . . might be called formal functionalism."

Albers, Donald J., et al., *Undergraduate Programs in the Mathematical and Computer Sciences: The* 1985-1986 *Survey*, MAA Notes Number 7, MAA, 1987; xx + 147 pp + 33 pp appendices, (P).

Results from a stratified random sample of schools offering undergraduate programs in mathematical or computer sciences. Some highlights: In 1985 mathematics enrollments at four-year colleges and universities were three times those in computer science and eight times those in statistics, but computer science graduated 50% more majors than mathematics and statistics combined. The full-time equivalent faculty in mathematics grew 6% since 1970, while course enrollments increased 42%; the percentage of doctorates among full-time faculty in mathematical and computer sciences dropped from 82% to 73% since 1975. Two-thirds of institutions with computer science majors require calculus for majors, one-half require matrix or linear algebra, and two-fifths require discrete mathematics. At two-year colleges, courses (in mathematics departments) in technical mathematics, business mathematics, and mathematics for liberal arts all declined 50% since 1980; remedial mathematics is now up to half of all enrollments in mathematical and computer sciences. Anyone trying to get a comprehensive view of undergraduate education in the United States needs to look at this report.

NEWS & LETTERS

LETTERS TO THE EDITOR

Dear Editor:

There is a small gap in the proof of the theorem in N.J. Lord's note "Matrices as sums of invertible matrices" (vol. 60, no. 1, pp. 33-35) at the end of which Lord deals with the case $q = 2$. Specifically, he wishes to exhibit a representation of the $n \times n$ matrices

$$\begin{bmatrix} \ell_k & 0 \\ 0 & 0 \end{bmatrix}$$

(where ℓ_k is the $k \times k$ identity) as a sum of invertible matrices with entries taken from the field of two elements. Lord does this by considering separately the cases k even and k odd. We read "If k is odd, it is enough to provide a splitting of ℓ_3 since we can then abut the splitting of ℓ_{k-3}." But doesn't this omit the case $k = 1$?

Fortunately, this is no difficulty since $n > 1$ so

$$\begin{bmatrix} 1 & 0 & \\ 0 & 0 & \\ & & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & \\ 1 & 0 & \\ & & \ell_{n-2} \end{bmatrix} + \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \ell_{n-2} \end{bmatrix}$$

is the representation we seek.

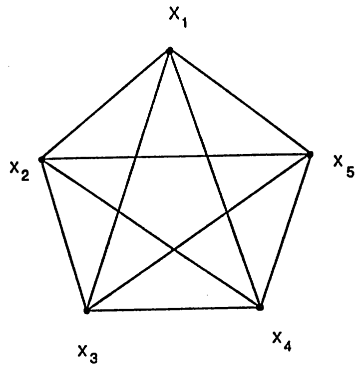
Daniel E. Otero
Pennsylvania State University
University Park, PA

Dear Editor:

I read your article providing solutions to the 27th International Math Olympiad problems. In particular, I was interested in the problem involving a regular pentagon. I had heard of the problem some months ago together with the comment that there is an elegant solution. I worked on the problem and came up with a solution that I thought must be the elegant one that had been referred to. Noticing that it was not any one of the three solutions included in your article, I am enclosing my solution for your information.

As you indicated, the key to the solution is to find an integer valued non-negative function whose value decreases when the given operation is performed. The function that I am going to use is the sum of the squares of the differences along the five diagonals. I have enclosed a short note indicating the proof.

Balakrishnan Krishnamurthy
Computer Research Laboratory
Tektronix, Inc.
Beaverton, Oregon 97077



Consider $S = (x_1-x_3)^2 + (x_2-x_4)^2 + (x_3-x_5)^2 + (x_4-x_1)^2 + (x_5-x_2)^2$

Let us apply the transformation around x_2 (assuming $x_2 < 0$):

$x_1 \to x_1 + x_2$

$x_2 \to -x_2$

$x_3 \to x_3 + x_2$

$x_4 \to x_4$

$x_5 \to x_5$

$S \to S' = (x_1-x_3)^2 + (x_2+x_4)^2 + (x_3-x_5+x_2)^2 + (x_4-x_1-x_2)^2 + (x_5+x_2)^2$

$S - S' = -2x_2(x_1+x_2+x_3+x_4+x_5)$

which is easily seen to be positive.

Dear Editor:

"Proof without Words: The Harmonic Mean-Geometric Mean-Arithmetic Mean-Root Mean Square Inequality," page 158 of the June 1987 issue, gives a very clear representation of part of the inequality in Figure 5 of our "Gini Means," *American Mathematical Monthly*, October 1986.

Notice that the fifth power mean, called the harmonic-root-mean, $\left[\frac{1}{2}(a^{-2} + b^{-2})\right]^{-1/2}$ can be added to the other four which are displayed. This mean is the distance from M to the unlabeled point of intersection of the line MR and the circle. The distance is less than HM.

By the way, the radius of the circle is the standard deviation of a and b. Also, the circle is the Appollonian circle of points M and H. (See *Excursions in Geometry* by C.S. Ogilvy, Oxford University Press, 1969, Chapter 2.)

David Farnsworth
Richard Orr
Department of Mathematics
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Dear Editor:

Professor Garza's interesting article in the June issue has an amusing slip-of-the-pen, p. 162, where it's asserted that 24^{24} is 'roughly a number with thirty-three zeros.' I think he meant that it is a number with thirty-four digits. (In fact a muMath calculation shows that 24^{24} = 1333735776850284124449081472843776, a number with only two zeros).

James Wendel
Department of Mathematics
The University of Michigan
Ann Arbor, MI 48109

Dear Editor:

In Vol. 60, No. 3 of this *MAGAZINE* in the article "Superexponentiation" is the statement "To the best of my knowledge, the $x \uparrow n$ function was first investigated in the early 1970's by A. Guyton."

In an NSF summer institute in 1963, Robert Bruce wrote a paper on the subject, in which he used the following references, among others.

1. L. Euler, De formulis exponentialibus replicatis, Acta Academiae Scientiarum Imperialis Petropolitanae, p. 38, 1777.

2. G. Eisenstein, Entwicklung von $a^{a^{a^{\cdots}}}$. Crelle's Journal 28, 49.

3. M.D. Grave, Sur les expressions dites surpuissances, Ann. de Math. 3rd Ser. 17 (1898), 80.

I guess the moral is: If you want to know who first considered something, check the works of Gauss and Euler.

E.R. Deal
Department of Mathematics
Colorado State University
Ft. Collins, CO 80523

P.S. Martin Gardner discussed the matter in the May 1973 issue of *Scientific American.*

Dear Editor:

An alert reader (Genji Yoshino) has noted that there is an error in one of the references in my article about pi in the June issue of the *MAGAZINE*. The correct reference is:

L. Baxter, Are π, e, and $\sqrt{2}$ equally difficult to compute?, *Amer. Math. Monthly* 88 (1981) 50-51.

An additional reference that has recently been published is:

J.M. Borwein and P.B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Complexity.* (Wiley, 1986).

This contains (among much other material) a complete explanation of the algorithms for pi depending on the arithmetic-geometric mean.

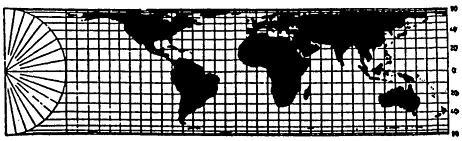
Gerald A. Edgar
Department of Mathematics
The Ohio State University
Columbus, OH 43210

Dear Editor:

While reading J.M. Sach's article on Mercator's map (this *MAGAZINE*, June 1987), I was reminded that Archimedes' beautiful demonstration that the area of a sphere and the lateral area of a circumscribing right cylinder are equal gives a rectangular area-preserving map.

The map is known to geographers as Lambert's equivalent cylindrical projection. ("Equivalent" refers to faithful representation of areas.) While searching for a good illustration to send you I found two U.S. Coast and Geodetic Survey Special Publications dealing with the subject. They are Number 236, General theory of equivalent projections by O.S. Adams (1945), and Number 245, Equal area projections for world statistical maps (1948) by F. Webster McBryde and Paul D. Thomas. The two publications may be of interest to *MAGAZINE* readers.

Howard E. Reinhardt
Dean and Professor of Mathematics
University of Montana
Missoula, MT 59812



Dear Editor:

Readers who enjoyed J.M. Sachs's "A curious mixture of maps, dates, and names" (60 (1987), 151-158) will no doubt also be interested in an instructional module on more details of the history and the mathematics of the Mercator projection. Entitled "Mercator's world map and the calculus," by Philip M. Tuchinsky, this 45-page module is UMAP Module 206, available from COMAP, Inc. (60 Lowell St., Arlington, MA 02174) for $2. The module was also reprinted in *UMAP Modules* 1977-1979: *Tools for Teaching*, pp. 677-727 (published by Birkhäuser but now out of print).

Paul J. Campbell
Beloit College
Editor, The UMAP Journal

NINETEENTH CANADIAN MATHEMATICAL OLYMPIAD (Problems)

1. Find all solutions of $a^2 + b^2 = n!$ for positive integers a, b and n with $a \leq b$ and $n < 14$.

2. The number 1987 can be written as a three-digit number xyz in some base b. If $x+y+z = 1+9+8+7$, determine all possible values of x, y, z and b.

3. Suppose $ABCD$ is a parallelogram and E is a point between B and C on the line BC. If the triangles DEC, BED and BAD are isosceles, what are the possible values for the angle DAB?

4. On a large flat field, n people are positioned so that for each person, the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd, show that there is at least one person left dry. Is this always true when n is even?

5. For every positive integer n, show that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}]$$

where $[x]$ is the greatest integer less than or equal to x (for example, $[2.3]=2$, $[\pi]=3$ and $[5]=5$).

SIXTEENTH U.S.A. MATHEMATICAL OLYMPIAD (Problems)

1. Determine all solutions in non-zero integers a and b of the equation $(a^2+b)(a+b^2) = (a-b)^3$.

2. AD, BE and CF are the bisectors of the interior angles of triangle ABC, with D, E and F lying on the perimeter. If angle EDF is 90 degrees, determine all possible values of angle BAC.

3. Construct a set S of polynomials inductively by the rules:

(i) $x \in S$;

(ii) if $f(x) \in S$, then $xf(x) \in S$ and $x+(1-x)f(x) \in S$.

Prove that there are no two distinct

Prove that there are no two distinct polynomials in S whose graphs intersect within the region $\{0<x<1\}$.

4. Three circles C_i are given in the plane. C_1 has diameter $AB=1$. C_2 is concentric and has diameter k where $1<k<3$. C_3 has center A and diameter $2k$. We regard k as fixed. Now consider all straight line segments XY which have one endpoint X on C_2, one endpoint Y on C_3 and contain the point B. For what ratio XB/BY will the segment XY have minimum length?

5. Given a sequence $x_1, x_2, \ldots, x_n$ of 0's and 1's, let A be the number of triples (x_i, x_j, x_k) with $i<j<k$ such that (x_i, x_j, x_k) equals (0,1,0) or (1,0,1). For $1 \leq i \leq n$, let d_i denote the number of j for which either $j<i$ and $x_j = x_i$ or else $j>i$ and $x_j \neq x_i$.

(a) Prove that $A = \binom{n}{3} - \binom{d_1}{2} - \binom{d_2}{2} - \cdots - \binom{d_n}{2}$.

(b) Given an odd number n, what is the maximum possible value of A?

INTERNATIONAL MATHEMATICAL OLYMPIAD TEAM RESULTS

In the 28th Annual International Mathematical Olympiad held in Havana, Cuba, in July, 1987, the United States team placed fifth with 220 points out of a possible 252. Two U.S. students, Jordon Ellenberg of Potomac, Maryland, and Eric Wepsic of Boston, Massachusetts, won gold medals for receiving a perfect score on the examination. Of the 237 students participating in the IMO from 42 countries, 22 students received the gold medal.

Three U.S. team members received second prizes: Robert Southworth of Winchester, Massachusetts, William Schneeberger of Oklahoma City, Oklahoma, and John Woo of Pepper Pike, Ohio. Matthew Cook of Evanston, Illinois, received a third prize.

In terms of total score, the top fifteen countries were as follows:

Romania	252
Federal Republic of Germany	248
USSR	235
German Democratic Republic	231
USA	220
Hungary	218
Bulgaria	210
China	200
Czechoslovakia	192
Great Britain	182
Vietnam	172
France	154
Austria	150
The Netherlands	146
Australia	143

****

SPECIAL ISSUE ON PÓLYA IN DECEMBER

George Pólya was born on December 13, 1887, and to mark the centenary, the entire December issue of MATHEMATICS MAGAZINE will be devoted to Pólya's work. Among others, there will be articles by Boas, Braden, Harary and Plochinski, Hilton and Pedersen, Kilpatrick, Lange, Read, Schattschneider, and Schoenfeld. A special problems section will include problems by Coxeter, Dudley, Erdős and Lovasz, Eves, Hillman, Klamkin, Knuth, L. Moser (submitted by W. Moser), Rosenbloom, and Seiffert.

****

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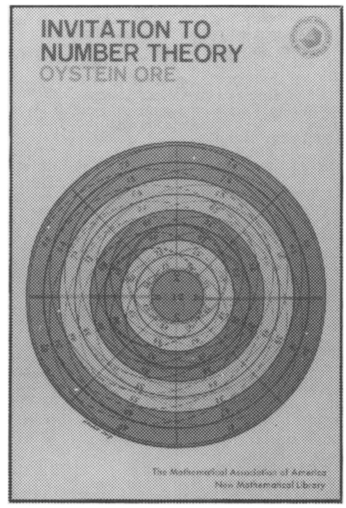

Invitation to Number Theory,
by Oystein Ore
129 pp., 1967, Paper, ISBN-0-88385-620-4
List: $9.90 MAA Member: $7.90

This outstanding book gives the reader some of the history of number theory, touching on triangular and pentagonal numbers, magic squares and Pythagorean triples, and numeration systems. It covers the primes and prime factorization (including the fundamental theorem of arithmetic), congruences (modular arithmetic) and their applications (including methods of checking numerical calculations), tests for primality, scheduling tournaments, and ways of determining the week day of a given date.

Ore writes of his book, "The purpose of this simple little guide will have been achieved if it should lead some of its readers to appreciate why the properties of numbers can be so fascinating. It would be better still if it would induce you to try to find some number relations of your own; new curiosities devised by young people turn up every year." The enterprise of making such discoveries is very broad including the invention and study of the curious sequence, 1, 11, 21, 1211, 111221, . . . (to understand it read it aloud) by Cambridge Professor John Horton Conway to the discovery of new Mersenne primes by high school students Laura Nickel and Kurt Noll. Ore's book is a good place for readers to learn of the fascination that numbers hold.

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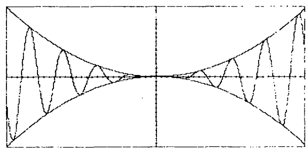
2. Definite integral of f(x). You can choose between Simpson's method or the trapezoid method after supplying the function to be integrated and the limits of integration. The number of sub-intervals to be used in the approximation is then typed in by the user as prompted.



$F(x) = x^2$ $F(x) = -x^2$ $F(x) = x^2 \sin(x)$

2. Numerical solutions of f(x) = 0. You can choose from the three methods; Newton's, Bisection, or False Positioning. Following simple prompts, you type in the functions, desired accuracy, and initial points.

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2. Solutions to m equations in n unknowns. The program finds the solutions to m linear equations in n unknowns. If a system is redundant or inconsistent, you see an appropriate message printed. (m,n≤20).

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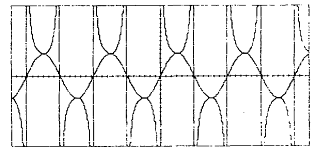

F(x) = sin(x) F(x) = csc(x)

Part III: Graphs of y = f(x)

This program draws the graphs of as many functions as desired on the same set of coordinate axes. You simply type in the function y = f(x) and choose horizontal and/or vertical scales following easy prompts. The illustration (above) shows the graphs of y = sin(x) and y = csc(x). This program is perfect for comparing the graphs of similar functions, e.g., f(x) = a sin(bx) for different values of a and b.

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The Mathematics of Games and Gambling,

by Edward Packel
141 pp., 1981, Paper, ISBN-0-88385-628-X
List: $11.00 MAA Member: $8.80

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MATHEMATICS MAGAZINE VOL. 60, NO. 4, OCTOBER 1987